A SYSTEM OF AXIOMS FOR GEOMETRY*

BY

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Contents.

CHAPTER I.

THE AXIOMS AND THEIR INDEPENDENCE.

§ 1. *Statement of the axioms.*

Euclidian geometry is a system of propositions codifying in a definite way† our spatial judgments. A system of axioms for this geometry is a finite number of these propositions satisfying the following two conditions:

1. Every proposition of euclidian geometry can be deduced from the axioms.
2. No axiom can be deduced from the other axioms.

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† By no means the only way. Cf. BOLYAI, LOBATCHEWSKY, VERONESE.

Thus an axiom differs from any other proposition of the science, as thus codified, in that it is *unproved.* *

The propositions brought forward as axioms in this paper are stated in terms of a class of elements called "points" and a relation among points called "order"; they thus follow the trend of development inaugurated by PASCH,† and continued by PEANO ‡ rather than that of HILBERT § or PIERI. ‖ ¶ All other geometrical concepts, such as line, plane, space, motion, are defined in terms of point and order. In particular, the congruence relations are made the subject of definitions ** rather than of axioms. This is accomplished by the aid of projective geometry according to the method first given analytically by CAYLEY and KLEIN. †† The terms "point" and "order" accordingly differ from the other terms of geometry in that they are *undefined.*

The axioms are twelve in number; they presuppose only the validity of the operations of logic and of counting (ordinal number). ‡‡

AXIOM I. There exist at least two distinct points.

AXIOM II. §§ If points A, B, C are in the order ABC, they are in the order CBA.

AXIOM III. If points A, B, C are in the order ABC, they are not in the order BCA.

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* Many writers replace the word "axiom" by "unproved proposition" or "primitive proposition."

† M. PASCH, *Vorlesungen über neuere Geometrie*, Leipzig, Teubner, 1882.

‡ G. PEANO, *I principii di geometria*, Turin, 1889; *Sui fondamenti della geometria*, Rivista di Matematica, vol. 4 (1894), pp. 51–59.

§ D. HILBERT, *Grundlagen der Geometrie*, Leipzig, 1899. The references are to the English translation by E. J. Townsend, Open Court Publishing Co., Chicago, 1902.

‖ M. PIERI, *Della geometria elementare come sistema ipotetico deduttivo. Monografia del punto e del moto*, Memorie della Reale Accademia delle Scienze di Torino (2), vol. 49 (1899), pp. 173–222. For other references to PIERI, see footnote to § 2, chap. III.

¶ Particular theorems or axioms derived from these and other writers I have acknowledged in footnotes or in §§ 3–14 of this chapter. I wish to express deep gratitude to Professor E. H. MOORE, who has advised me constantly and valuably in the preparation of this paper, and also to Messrs. N. J. LENNES and R. L. MOORE, who have critically read parts of the manuscript.

** A definition is an agreement to substitute a simple term or symbol for more complex terms or symbols. On the distinction between definitions and axioms see A. PADOA, Bibliothèque du Congrès International de Philosophie, III, "Logique et Histoire des Sciences" (1901), p. 309; PEANO, ibid., p. 279; E. H. MOORE, Bulletin of the American Mathematical Society (2), vol. 9 (1903), p. 402. O. VEBLEN, Monist, vol. 13, no. 2 (January, 1903), pp. 303–9.

†† A. CAYLEY, *Sixth Memoir on Quantics*, Collected Works, vol. 2, p. 56 (see also CAYLEY's note, Ibid., p. 605).

F. KLEIN, Mathematische Annalen, vol. 4 (1870), p. 573.

J. THOMAE, *Geometrie der Lage*, Halle, 1873.

E. B. WILSON, *Projective and Metric Geometry*, Annals of Mathematics, vol. 5, p. 145 (1904).

‡‡ In particular, the distinctness of A and B in the order ABC is a theorem.

§§ Mr. R. L. MOORE suggests that "If A is a point, B is a point, C is a point" would be a more rigorous terminology for the hypotheses of II, III, IV, inasmuch as we do not wish to imply that any two of the points are distinct.

AXIOM IV. If points A, B, C are in the order ABC, then A is distinct from C.

AXIOM V. If A and B are any two distinct points, there exists a point C such that A, B, C are in the order ABC.

DEF. 1. The *line* AB ($A \neq B$) consists of A and B and all points X in one of the possible orders ABX, AXB, XAB. The points X in the order AXB constitute the *segment* AB. A and B are the *end-points* of the segment.

AXIOM VI. If points C and D ($C \neq D$) lie on the line AB, then A lies on the line CD.

AXIOM VII. If there exist three distinct points, there exist three points A, B, C not in any of the orders ABC, BCA, or CAB.

DEF. 2. Three distinct points not lying on the same line are the *vertices* of a *triangle* ABC, whose *sides* are the segments AB, BC, CA, and whose *boundary* consists of its vertices and the points of its sides.

AXIOM VIII. If three distinct points A, B, and C do not lie on the same line, and D and E are two points in the orders BCD and CEA, then a point F exists in the order AFB and such that D, E, F lie on the same line.

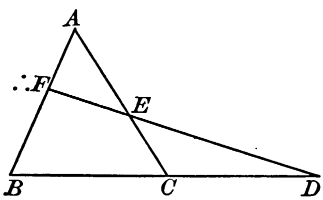

FIG.

DEF. 5. A point O is *in the interior of* a triangle if it lies on a segment, the end-points of which are points of different sides of the triangle. The set of such points O is *the interior* of the triangle.

DEF. 6. If A, B, C form a triangle, the *plane* ABC consists of all points collinear with any two points of the sides of the triangle.

AXIOM IX. If there exist three points not lying in the same line, there exists a plane ABC such that there is a point D not lying in the plane ABC.

DEF. 7. If A, B, C, and D are four points not lying in the same plane, they form a *tetrahedron* $ABCD$ whose *faces* are the interiors of the triangles ABC, BCD, CDA, DAB (if the triangles exist)* whose *vertices* are the four points, A, B, C, and D, and whose *edges* are the segments AB, BC, CD, DA, AC, BD. The points of faces, edges, and vertices constitute the *surface* of the tetrahedron.

DEF. 8. If A, B, C, D are the vertices of a tetrahedron, the space $ABCD$ consists of all points collinear with any two points of the faces of the tetrahedron.

* Without this phrase, the definition might be thought to involve some hypothesis about the non-collinearity of the vertices of the tetrahedron. In K'_{III} (§ 12) there is a tetrahedron of only three faces.

AXIOM X. If there exist four points neither lying in the same line nor lying in the same plane, there exists a space $ABCD$ such that there is no point E not collinear with two points of the space, $ABCD$.

AXIOM XI. If there exists an infinitude of points, there exists a certain pair of points AC such that if $[\sigma]$* is any infinite set of segments of the line AC, having the property that each point which is A, C or a point of the segment AC is a point of a segment σ, then there is a finite subset $\sigma_1, \sigma_2, \cdots, \sigma_n$ with the same property.

AXIOM XII. If a is any line of any plane α there is some point C of α through which there is not more than one line of the plane α which does not intersect a.

§ 2. *Categorical system. Independence of axioms in general.*

Inasmuch as the terms *point* and *order* are undefined one has a right, in thinking of the propositions, to apply the terms in connection with any class of objects of which the axioms are valid propositions. It is part of our purpose however to show that there is *essentially only one* class of which the twelve axioms are valid. In more exact language, any two classes K and K' of objects that satisfy the twelve axioms are capable of a one-to-one correspondence such that if any three elements A, B, C of K are in the order ABC, the corresponding elements of K' are also in the order ABC. Consequently any proposition which can be made in terms of points and order either is in contradiction with our axioms or is equally true of all classes that verify our axioms. The validity of any possible statement in these terms is therefore completely determined by the axioms; and so any further axiom would have to be considered redundant.† Thus, if our axioms are valid geometrical propositions, they are sufficient for the complete determination of euclidian geometry.

A system of axioms such as we have described is called *categorical*, whereas one to which it is possible to add independent axioms (and which therefore leaves more than one possibility open) is called *disjunctive*.‡ The categorical property of a system of propositions is referred to by HILBERT in his "Axiom der Vollständigkeit," which is translated by TOWNSEND into "Axiom of Completeness." E. V. HUNTINGTON,§ in his article on the postulates of the real number system, expresses this conception by saying that his postulates are sufficient for the *complete definition* of essentially a single assemblage. It would probably be better to reserve the word *definition* for the substitution of one

* $[e]$ denotes a set or class of elements, any one of which is denoted by e alone or with an index or subscript.

† Even were it not deducible from the axioms by a finite number of syllogisms.

‡ These terms were suggested by PROFESSOR JOHN DEWEY.

§ E. V. HUNTINGTON, *A Complete Set of Postulates for the Theory of Absolute Continuous Magnitude*, Transactions of the American Mathematical Society, vol. 3, p. 364 (1902).

symbol for another, and to say that a system of axioms is categorical if it is sufficient for the complete *determination* of a class of objects or elements.

An example of a disjunctive set of axioms is furnished as soon as we leave out any one of the axioms in the set above. For instance, all the first eleven axioms are satisfied if the class of *points* is taken to consist of all the points interior to a sphere and *order* as referring to the usual relation of order among these points. Axiom XII is evidently not a valid proposition of this class. The first eleven axioms (excluding XII) therefore can refer to at least two essentially different classes and so form a disjunctive system.

The example just given serves also to show that axiom XII satisfies the second of the conditions named in § 1 for a proposition to be an axiom. It cannot be logically deduced from axioms I–XI; in technical language, it is *independent*.

In the following sections, besides some remarks on the history and formal peculiarities of the axioms, the independence of each of the twelve axioms is to be proved in the manner just indicated. That is, for axiom $N (N = \mathrm{I}, \mathrm{II}, \cdots, \mathrm{XII})$ a class of objects K_N will be exhibited which satisfies each of the axioms except N and for which N is not satisfied. Therefore if axiom N were omitted the system of axioms would be disjunctive and so the validity of N is not determined by the other axioms.

§ 3. *Parallels. Axiom XII.*

The parallel axiom was the first whose independence was seriously studied. Its history* is so well known that I need only mention the names, BOLYAI, LOBATCHEWSKY, RIEMANN, BELTRAMI, CAYLEY, KLEIN. The form in which it is here stated is due to C. BURALI-FORTI.†

The class of points interior to a sphere has been given in § 2 as K_{XII}.

The questions whether axiom XII would still be sufficient if stated for only one plane instead of for any plane, or for only one line of any plane instead of for any line of any plane can both be answered in the negative. The class K'_{XII}, consisting of all the points on a certain side of a certain plane π, is such that in at least one plane, namely, any plane parallel to π, there is only a single parallel to any line through any point. In the same system, in any plane, there is at least one line, namely, the line parallel to π, to which there is only one parallel through any point.

§ 4. *Continuity. Axiom XI.*

The proposition here adopted as the continuity axiom is referred to by

* For bibliography, see G. B. HALSTED, American Journal of Mathematics, vol. 7, p. 264 (1878); also American Mathematical Monthly, vol. 8, pp. 216–230 (1901).

† C. BURALI-FORTI, *Les postulats pour la géométrie d'Euclide et de Lobatchewsky*, p. 264, Verhandlungen des Ersten Internationalen Mathematiker Congresses, Leipzig, 1898.

SCHOENFLIES * as the HEINE-BOREL theorem. So far as I know, it was first stated formally (as a theorem of analysis rather than of geometry) by BOREL † in 1895 but is involved in the proof of the theorem of uniform continuity given by HEINE ‡ in 1871. The idea of its equivalence with the Dedekind cut axiom was the result of a conversation with Mr. N. J. LENNES.

The independence of XI is well known. XI and its consequence, the so-called Archimedean axiom, § have been discussed by VERONESE,|| STOLZ,¶ LEVI-CIVITA, ** HILBERT, †† and DEHN. ‡‡

That theorem 38, which shows the existence of at least one parallel to a given line, cannot be proved without recourse to XI is shown by K_{XI} of which the points are the rational points of an ordinary coördinate geometry, viz., points whose coördinates are all rational numbers, and also the points at infinity collinear with at least two rational points. To define order, consider the plane σ through the points $(\pi, 0, 0)$, $(0, \pi^2, 0)$, $(0, 0, \pi^3)$ where π is the ludolphian constant. This plane contains no point of K_{XI}. *Points A, B, C, of K_{XI} are in the order ABC if, in the ordinary geometry, A, B. C are collinear and A, C are separated by B and the point in which the line AB meets the plane σ.* The axioms I–X are satisfied and every two complanar lines meet.

K_{XI} is determined categorically by a system of twelve axioms, the first ten of which are identical with I–X, and the eleventh of which is

XI′. On a certain line there are three points that are harmonically related to every point of the line.

and the twelfth of which is

XII′. Every two complanar lines meet in a point.

On the basis of these axioms the same proof of the fundamental theorem of pro-

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* A. SCHOENFLIES, *Bericht über die Mengenlehre*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 8 (1898), p. 51.

† É. BOREL, Annales de l'École Normale Supérieure (3), vol. 12 (1895), p. 51. It is stated by BOREL for a numerable set of segments.

‡ E. HEINE, *Die Elemente der Functionenlehre*, Crelle, vol. 74 (1872), p. 188.

§ Cf. O. STOLZ, *Zur Geometrie der Alten, insbesondere über ein Axiom des Archimedes*, Mathematische Annalen, vol. 22 (1883).

|| G. VERONESE, *Fondamenti di geometria, a più dimensioni e a più specie di unità rettilinee*, Padua, 1891. German translation by Schepp, Leipzig, 1894.

¶ O. STOLZ, *Ueber das Axiom des Archimedes*, Mathematische Annalen, vol. 39 (1891), p. 107.

** T. LEVI-CIVITA, Memorie della Reale Accademia dei Lincei Roma, vol. 7 (1898), pp. 91–96, 113–131.

†† D. HILBERT, l. c., § 12 and also, *Ueber den Satz von der Gleichheit der Basiswinkel im gleichschenkligen Dreieck*, Proceedings of the London Mathematical Society, vol. 35 (1902), p. 50.

‡‡ M. DEHN, *Die Legendre'schen Sätze über die Winkelsumme im Dreieck*, Mathematische Annalen, vol. 53 (1900), p. 404.

§§ A test of completeness for the axiomatic analysis of a mathematical science would be the question whether for every theorem one can specify a subset of the axioms, sufficient and individually indispensable for its proof. In the present system the answer to this question is affirmative in the cases that I have investigated.

jective geometry can be made as in § 6, chapter III because theorem 67 is an evident consequence of XI′.

§ 5. *Tridimensionality. Axiom* X.

As K_{X} may be taken the usual four-dimensional geometry. The axiom equivalent to X in the systems of PASCH and HILBERT is:

"If two planes α, β have a point A in common, then they have at least a second point B in common." *

This is essentially theorem 25, § 3, chapter II. PEANO's axiom XVI in the *Principii* is nearly the same as ours; namely, "if A and B are such that the segment AB meets a plane π, then for any point X, either the segment AX or the segment BX meets π."

§ 6. *Relation of plane and space geometry. Axiom* IX.

The class of points K_{IX} is the set of points in an ordinary plane subject to the usual order relations. The non-desarguesian "geometries" of HILBERT † and MOULTON ‡ may be enumerated respectively as K_{IX}^{H} and K_{IX}^{M}. They show the essential dependence of the Desargues theorem (62, chap. III) and harmonic properties (§ 5, chap. III) of the plane upon axiom IX.

This axiom, not stated explicitly by PASCH, is number XV (page 23) in PEANO's *Principii*, and is part of I 7 in HILBERT's system. For the suggestion to state this axiom for only a *single* plane as well as VII for only a single triad of points I am indebted to Professor MOORE.

§ 7. *Significance and construction of finite independence proofs.*

The independence proofs of the first eight axioms can all be exhibited by means of finite classes of elements or points. That such independence proofs are not possible for IX–XII is a consequence of theorem 12, based on I–VIII, that every line contains an infinitude of points. The finiteness of $K_{\mathrm{I}}, \cdots, K_{\mathrm{VIII}}$ shows conversely that every one of the first eight axioms is essential to the proof of theorem 12. It also brings into relief an important difference between our system of axioms and the systems based on RIEMANN's concept of space as fundamentally a number-manifold. By the method of RIEMANN, the axioms serve to distinguish space from all possible number-manifolds, by the other method they distinguish it from all systems of objects of any kind.

* The form quoted is HILBERT's II 6.

† D. HILBERT, *Foundations of Geometry*, p. 74, § 23.

‡ F. R. MOULTON, *A simple non-desarguesian plane geometry*, Transactions of the American Mathematical Society, vol. 3 (1902), p. 192.

The description of the following tactical systems is preliminary to the exposition of $K_{\mathrm{I}}, \cdots, K_{\mathrm{VIII}}$.*

By $S\{3, 2, 5\}$ is denoted the system of ordered triads or permutations:

123, 234, 345, 451, 512	135, 352, 524, 241, 413
321, 432, 543, 154, 215	531, 253, 425, 142, 314

Each ordered pair of elements determines a unique element that precedes it, a unique element that follows it and a unique middle element. All the triads can be obtained from 123, 321, 531, 135 by the cyclic substitution (12345).

By $S[3, 2, 7]$ is denoted the *triadic system* of seven vertically printed triads:

0	1	2	3	4	5	6
1	2	3	4	5	6	0
3	4	5	6	0	1	2

It has the property 1st that every pair of elements lies in a unique triad and 2nd every pair of triads has a unique common element. $S[3, 2, 7]$ is invariant under the well-known group G^7_{168} of order 168 with generators,

$$S_1 = (0123456),$$
$$S_2 = (0)(124)(365),$$
$$S_3 = (0)(1)(3)(26)(45),$$
$$S_4 = (0)(1)(3)(25)(46).$$

All the triads are obtained from any one by the cyclic substitution S_1. Notation apart, this is the only triadic system in seven elements.

Analogous to a triadic system is the pentadic system $S[5, 2, 21]$ of twenty-one vertically printed pentads:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0
6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3	4	5
8	9	10	11	12	13	14	15	16	17	18	19	20	0	1	2	3	4	5	6	7
18	19	20	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

Every pair of elements lies in one and only one pentad and every two pentads have in common one and only one element. $S[5, 2, 21]$ is invariant under a group G^{21} of order $21 \cdot 20 \;\; 16 \cdot 9 \cdot 2 = 120960$, with the following generators:

* A rich collection of such systems (not, however, including $S[5, 2, 21]$) is to be found in the memoir, *Tactical Memoranda*, I–III, by E. H. MOORE, American Journal of Mathematics, vol. 18 (1896), p. 264.

$T_1 = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20)$,

$T_2 = (0)(1, 2, 4, 8, 16, 11)(3, 6, 12)(5, 10, 20, 19, 17, 13)(7, 14)(9, 18, 15)$,

$T_3 = (0)(1)(2, 9)(7, 19)(6, 8, 18)(12, 11, 16, 4, 15, 3)(5, 14, 17, 10, 20, 13)$,

$T_4 = (0)(1)(3, 16)(17, 13)(6, 18, 8)(2, 4, 15, 9, 12, 11)(7, 10, 20, 19, 5, 14)$,

$T_5 = (0)(1)(2)(6)(8)(18)(7, 19, 9)(12, 16, 15)(3, 20, 10)(4, 17, 14)(5, 13, 11)$,

$T_6 = (0)(1)(2)(3)(8)(9)(16)(12, 15)(13, 17)(6, 18)(7, 19)(5, 14)(4, 11)(10, 20)$.

All the pentads are obtained from any one by the cyclic substitution T_1. The group is doubly transitive and indeed such that any three elements not belonging to the same pentad can be transformed into $0, 1, 2$.

§ 8. *Triangle transversal. Axiom* VIII.

Axiom VIII was originally stated by PASCH* in a form equivalent to this: "A line lying wholly in a plane and intersecting one side of a triangle of the same plane intersects the perimeter of the triangle in one other point." It was retained by HILBERT as his II 5 in practically the same form. PEANO† made of it two axioms (XIII and XIV) from either of which VIII can be deduced. The statement of the axiom was made considerably weaker than any of its previous forms by Professor MOORE,‡ and the present form is weaker than his.

As the points of K_{VIII} we take the twenty-one elements of $S[5, 2, 21]$. The pentad $0, 1, 6, 8, 18$ taken in a definite order determines a system of triads of the type $S\{3, 2, 5\}$. The order of any pentad being defined as that obtained from the above by a cyclic substitution, the twenty-one pentads determine forty-two systems of triads—420 triads in all.

Three points of K_8 are in the order ABC if ABC is one of the 420 *triads just specified.*

Axiom I is valid for K_{VIII} because there are twenty-one points, and axiom II because the order ABC implies a system $S\{3, 2, 5\}$ which by its definition includes the triad CBA. Axiom III is verified because each triad, and indeed each pair, appears in only one pentad and that pentad determines only one order system of the type $S\{3, 2, 5\}$ in which each triad can appear only once. IV is equivalent to the statement that each triad consists of distinct elements. To verify V observe that every pair of elements determines a pentad, and in each pentad by the definition of $S\{3, 2, 5\}$ each pair of elements has a predecessor and a successor.

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* M. PASCH, l. c., p. 21.

† G. PEANO, l. c., pp. 18, 21.

‡ E. H. MOORE, *On the projective axioms of geometry*, Transactions of the American Mathematical Society, vol. 3 (1902), p. 147.

A line in $K_{\rm VIII}$ consists of the points of a pentad. VI is a consequence of the proposition that any pair of elements is contained in only one pentad. VII is verified by the points $0, 1, 2$ which appear together in no pentad. Since only one point is between any two points, each side of a triangle in $K_{\rm VIII}$ consists of but one point, and thus a plane consists of at most $3 + 3 \times 3 = 12$ points. This verifies IX. X and XII are consequences of the proposition that $0, 1, 2$ can be transformed into any non-collinear triad and that every two lines have a point in common. XI is satisfied *vacuously*, i. e., its hypothesis is not fulfilled.

To show that VIII is contradicted, consider the triangle $A = 0, B = 1, C = 3$. Then $D = 13$ and $E = 9$. The point of intersection F of the lines DE and AB is 18, and hence A, B, and F are in the order FAB.

§ 9. *Axiom* VII.

In the class $K_{\rm VII}$ there are five points, $1, 2, 3, 4, 5$. *Points A, B, C are in the order ABC if and only if they constitute a triad ABC in the system* $S\{3, 2, 5\}$. The verification of I–VI for $K_{\rm VII}$ was in effect carried out in § 8. VII is negated since all the five points are collinear. VIII–XII are satisfied vacuously.

As $K'_{\rm VII}$ one may take the points of a geometrical line with the usual relations of order.

§ 10. *Axiom* VI.

In the class $K_{\rm VI}$ there are twenty-six points $P_1, P_2, P_3, P_4, P_5, 0, 1, 2, \cdots, 20$. *Points ABC are in the order ABC if and only if they constitute a triad in one of the sets of triads,*

P_1, P_2, P_3, P_4, P_5 *ordered according to* $S\{3, 2, 5\}$,

$0, 1, \cdots, 20$ *ordered according to* $S[5, 2, 21]$ *as in* § 8,

$$P_i h P_j \qquad (i, j = 1, \cdots, 5;\ i \neq j;\ h = 0, \cdots, 20),$$

$$h P_i k \qquad (h, k = 0, \cdots, 20;\ h \neq k;\ i = 1, \cdots, 5).$$

VI is contradicted in $K_{\rm VI}$ because $K_{\rm VI}$ includes the orders $0P_1 1$ and $1P_1 2$, without containing 012, 120, or 201. Since all the twenty-six points lie on the line P_1P_2 the hypotheses of VIII–XII are not fulfilled. I–V are easily verified.

§ 10. *Axiom* V.

$K_{\rm V}$ consists of the points, 1 and 2, with the agreement that *points A, B, C are in the order ABC if and only if $A \neq B$, $B \neq C$, $C \neq A$.* I is evidently verified and since there exists no triple of points in the order ABC, V is denied. All the other axioms are satisfied vacuously.

§ 11. *Axiom* IV.

K_{IV} consists of two points, 1 and 2 with the agreement that *points* A, B, C *are in the order* ABC *if* $A = 1$, $B = 2$, $C = 1$, *or if* $A = 2$, $B = 1$, $C = 2$. IV is evidently contradicted, and I, II, III, V, VI are evidently satisfied. VII–XII are satisfied vacuously.

§ 12. *Axiom* III.

K_{III} consists of two points, 1 and 2 with the agreement that *points* A, B, C *are in the order* ABC *if* $A \neq B$ *and* $B = C$ *or if* $A = B$ *and* $B \neq C$.

K'_{III} consists of seven points $0, 1, 2, 3, 4, 5, 6$, with the agreement that points A, B, C are in the order ABC if they are distinct and constitute a triad of the triadic system $S[3, 2, 7]$ (§ 7). In verifying K'_{III} note that a line is a triad of points. The plane 123 consists of the collinear points $4, 5, 0$. The tetrahedron 0124 has as faces the interiors of the triangles 012, 024, 041 (not 124) consisting of the triads 364, 165, 325. Thus the space 0124 includes all seven points.

K''_3 consists of the points of the ordinary projective geometry (see chapter III) with the agreement that points A, B, C are in the order ABC if they are distinct and incident with the same projective line.

§ 13. *Axiom* II.

K_2 consists of two points, 1 and 2, with the agreement that *points* A, B, C *are in the order* ABC *if and only if* $A \neq B$ *and* $B = C$.

§ 14. *Axiom* I.

K_1 consists of one point with the agreement that *points* A, B, C *are in the order* ABC *if and only if* $A \neq B \neq C \neq A$. The other axioms are satisfied vacuously.

CHAPTER II.

General properties of space.

§ 1. *Properties of the line.*

Axiom I. *There exist at least two distinct points.*

Axiom II. *If points* A, B, C *are in the order* ABC, *they are in the order* CBA.

Axiom III. *If points* A, B, C *are in the order* ABC, *they are not in the order* BCA.

Th. 1 (II, III).* If points A, B, C are in the order ABC, they are not in the order CAB.

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* The roman numbers in parentheses denote the axioms upon which the proof of the theorem depends. Theorems are occasionally referred to by arabic numbers in parentheses.

Proof. Since the points A, B, C are in the order ABC, that is, for brevity, ABC, then by II CBA, and hence by III the order BAC cannot exist, and therefore by II CAB cannot.

TH. 2 (II, III). The order ABC implies that A is distinct from B and B from C.

Proof. If A were the same as B, the orders ABC and BAC would be the same, contrary to theorem 1. Therefore A is distinct from B. On account of II, the same proof shows that B is distinct from C.

AXIOM IV. *If points A, B, C are in the order ABC, then A is distinct from C.*

Corollary (II, III, IV). The order ABC implies that A, B, and C are distinct points.

AXIOM V. *If A and B are any two distinct points, there exists a point C such that A, B, and C are in the order ABC.*

If A and B stand for the same point we write $A = B$; if for distinct points $A \neq B$.

DEF. 1. The *line* AB ($A \neq B$) consists of A and B and all points X in any one of the orders ABX, AXB, XAB. The points X in the order AXB constitute the *linear segment* AB, or *segment* AB. They are said to lie *between* A and B to be *points of the segment* AB or *interior points* of the segment AB; A and B are *end-points* of the segment. The points X in the order ABX constitute the *prolongation* of the segment AB beyond B, and those in the order XAB the prolongation beyond A. The points of a line are said to *lie on* the line.

In terms of def. 1 the results of the preceding axioms and theorems are:

TH. 3 (I, II, II, IV, V). Every pair of distinct points A and B defines one and only one line AB, and one and only one segment AB. The line or the segment AB is the same as the line or the segment BA. A and B are not points of the segment AB. No two of the three sets of points that (with A and B) constitute the line AB (namely, the segment AB and its two prolongations), have a point in common. There exists at least one point on each prolongation of the segment AB.

It has not been shown as yet that every segment AB contains a point nor that the pair of points AB may not lie on some other line than the line AB.

AXIOM VI. *If points C and D ($C \neq D$) lie on the line AB, then A lies on the line CD.*

TH. 4 (I–VI). Two distinct points lie on one and only one line.

Proof. By theorem 3, it is necessary only to show that if C and D are any two distinct points of a line AB the line CD is the same as the line AB. Suppose first that D is the same as B; we have to show that the line AB is the same as the line BC. Let X be any point of the line BC ($A \neq X \neq B$).

As A is known to be a point of BC, it follows from VI that B is a point of the line AX. Hence we have one of the orders ABX, AXB or XAB, and therefore X is a point of AB. Similarly any point of AB is shown to be a point of BC. Hence the lines AB and BC are the same. Similar proofs hold if $D = A$ or $C = A$ or $C = B$.

If D and C are both distinct from A and B, we argue that the line AB is the same as BC, which is the same as CD.

Corollary. Two distinct lines can have in common at most one point. Such a common point F is called a point of intersection and the lines are said to *intersect* in F.

AXIOM VII. *If there exist three distinct points there exist three points A, B, C, not in any of the orders ABC, BCA, or CAB.*

TH. 5 (I–VII). If DE is any line, there exists a point F not lying in this line.

Proof. If the line DE contains every point, it would contain the three points A, B, C mentioned in VII. The line AB, by theorem 4, would be the same as DE, and hence AB would contain the point C, contrary to VIII.

DEF. 2. Points lying in the same line are said to be *collinear*. Three distinct non-collinear points, A, B, and C, are the *vertices* of a *triangle* ABC, whose *sides* are the segments AB, BC, CA, and whose *boundary* consists of its vertices and the points on its sides.

AXIOM VIII (*Triangle transversal axiom*). *If three distinct points A, B, C do not lie on the same line, and D and E are two points in the order BCD and CEA, then a point F exists in the order AFB, and such that D, E, F lie on the same line.*

TH. 6 (I–VIII). Between every two distinct points there is a third point.

Proof. Let A and B be the given points (fig. 2). By theorem 5 there is a point E not lying on the line AB. By V points C and D exist, satisfying the order-relations AEC and BCD. Hence, by VIII, F exists in the order AFB.

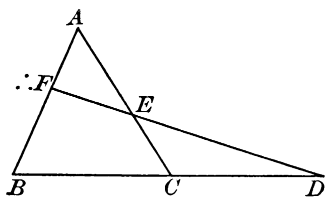


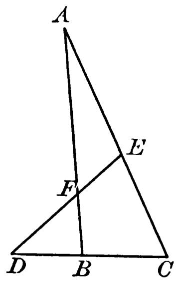
FIG. 2.

TH. 7 (I–VI, VIII). The points D, E, F, of axiom VIII, are distinct, and lie in the order DEF.

Proof. If D, E, F were not distinct, we should by axiom VI have A, B, C collinear. Hence there must be one of the three orders DEF, EFD, FDE, and it is necessary to show EFD and FDE impossible.

1st. Suppose the points were in the order EFD (fig. 3). Consider the triangle DEC (D, E, C being non-collinear by VI), and the orders AEC, DFE. By VIII, a point X exists on the line AF and in the order DXC. But as this point is common to the lines AF and DC, which have B in common, by the corollary of theorem 4, $X = B$. Hence we have the order DBC contrary to hypothesis. Hence DFE is impossible.

2d. Suppose EDF (fig. 4). Consider the triangle EFA (E, F, A being non-collinear by VI) and the orders AEC and EDF. By the same argument as above, we find that B is on the line CD and in the order ABF, which contradicts our hypothesis. Hence EDF is impossible.



·FIG. 3.

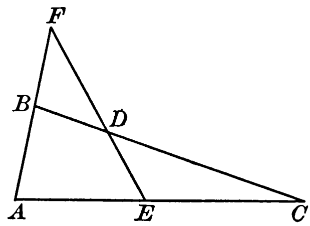


FIG. 4.

TH. 8 (I–VI, VIII). If points A, B, C form a triangle, there is no line that has a point in common with each of the sides AB, BC, CA.

Proof. If the theorem is not verified, we have three collinear points A', B', C', in the three orders $AB'C$, $BC'A$, and $CA'B$. It is evidently sufficient to show the order $A'B'C'$ impossible.

If $A'B'C'$, then A', C', and B would form a triangle; for otherwise A, B, C would be collinear. In connection with the triangle $A'C'B$, consider the order relations $BA'C$, $A'B'C'$. By theorem 7, a point X exists in the order BXC', and $CB'X$. Hence X is the intersection of the lines AB and $B'C$. Hence $X = A$. Hence BAC' contrary to hypothesis.

DEF. 3. If points A, B, C, D are in the orders ABC, ABD, ACD, BCD, they are said to be in the order $ABCD$.

TH. 9 (I–VIII).* To any four distinct points of a line, the notation A, B, C, D, may always be assigned so that they are in the order $ABCD$.

Proof. The proof of this theorem depends on the following six lemmas:

1. If ABC and BCD, then ABD.

By V and theorem 5, there exist points P and O not in the line AB but in the order BPO (Fig. 5). By VIII and theorem 7, since we have the triangle OBC and the orders ABC and BPO there exists a point Q in the orders OQC and APQ. Like-

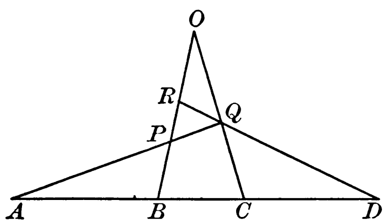


FIG. 5.

* This proposition was given as "Axiom II 4" in HILBERT's *Festschrift* (p. 7). Its redundancy as an axiom of HILBERT's system I, II was proved by E. H. MOORE, *On the projective axioms of geometry*, Transactions of the American Mathematical Society, January, 1902, vol. 3, p. 142–168, 501. A second proof has been given by Mr. R. L. MOORE, cf. p. 98, American Mathematical Monthly, April, 1902.

wise, since we have the triangle OBC and the orders BCD and OQC, there exists a point R in the orders ORB and DQR. Finally, since we have the triangle AQD, and the orders RQD and APQ, there exists a point X in the orders AXD and RPX. But the intersection of the lines AD and RP is B. Hence by the corollary to theorem 4, $X = B$. Hence ABD.

2. If ABC and ABD then either BCD or BDC.

In consequence of VI it is only necessary to show the order CBD impossible. By (5) and V there exist points O and P not collinear with B and C and in the order OCP (fig. 6). Therefore unless our theorem is valid, we have a triangle OCD and the orders OCP and CBD. Hence by VIII and (7), there exists a point Q in the orders OQD and PBQ. Now we have the tri-

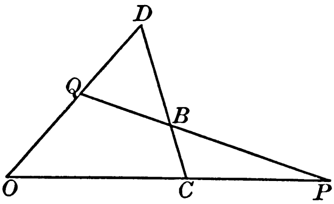


FIG. 6.

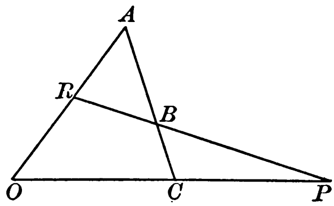


FIG. 7.

angle ACO (fig. 7) and the orders OCP and ABC. Hence a point R exists in the orders PBR and ARO. From the orders PBR and PBQ it follows by VI that R, B, and Q are collinear. But R, B, and Q are points of the sides OA, AD, and DO, of the triangle OAD, and this is in contradiction with theorem 8. Hence CBD is impossible.

3. It follows from 1 and 2 that if ABC and ABD, then either ACD or ADC.

4. If ABD and ACD, then either ABC or ACB.

By theorem 4, we must have either ABC or ACB or BAC. In the last case, from BAC and ACD we could by 1 argue BAD, contrary to hypothesis

5. If ABC and ACD, then BCD.

By theorem 4, if not BCD then BDC or CBD. In the first case, DCA leads by lemma 1 to BCA—a contradiction; and in the second case, CBA leads by lemma 3 to CAD or CDA, either of which contradicts ACD.

6. If ABC and ACD, then ABD.

By lemma 5 we have BCD, and hence ABC leads by lemma 1 to ABD.

We are now ready to complete the proof of theorem 9. Assign the notation A, B, C to three of the points so that they are in the order ABC. The fourth point D will either lie in the order

a) DAB, in which case it follows from 1 that we have $DABC$.

b) ADB, in which case it follows from 5, 6 that we have $ADBC$.

c) ABD, in which case it follows from 2, 1 that we have $ABCD$ or $ABDC$.

In any case the notation can be assigned so that the order is $ABCD$.

DEF. 4. The order $A_1 A_2 \cdots A_l \cdots A_n$ means that the points A_i, A_j, A_k are in the order $A_i A_j A_k$ if $i < j < k$ $(i, j, k = 1, 2, \cdots, n)$.

TH. 10 (I–VIII). To any finite number n, of distinct points of a line, the notation can be so assigned that they are in the order $A_1 A_2 \cdots A_l \cdots A_n$.

Proof. By mathematical induction.

TH. 11 (I–VIII). Any finite set of n distinct points separates a line into $n-1$ segments and two prolongations of segments, no two of which have a point in common.

Proof. It is a consequence of theorem 3 that one point separates a line into two mutually exclusive parts, and two points into three mutually exclusive parts. One point separates a segment into two mutually exclusive segments (9); and a prolongation into a segment and a prolongation. Hence if m points separate a line into $m-1$ segments and two prolongations, adding one point adds one segment, and therefore $m+1$ points divide a line into m segments and two prolongations. Hence by induction from the case $m = 2$ the theorem is proved.

TH. 12 (I–VIII). Between any two points A and B $(A \neq B)$ and also on either prolongation of the segment AB, there exist an infinitude of points.

Proof.* By theorems 6 and 9 it can be proved that if n is any integer there are more than n points X in the order AXB or ABX.

§ 2. *Properties of the plane.*

TH. 13 (I–VIII).† If A, B, and C form a triangle, and D and F exist in the orders BCD and AFB, then E exists in the orders AEC and DEF.

Proof (fig. 8). By V and (9) there exists a point O in the order $OBFA$. Hence by VIII and (7), from the triangle DBF and the orders OBF and BCD, we obtain a point P in the orders OCP and FPD, and from the triangle DAF and the orders OFA, FPD a point Q in the orders OPQ and AQD. By (9), we then have $OCPQ$. Hence in the triangle CAQ we have AQD and CPQ. Hence E exists in the order AEC, and on the line PD, whence by (7) E is in the order DEF.

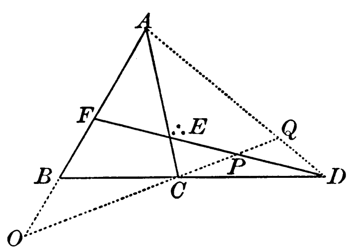


FIG. 8.

TH. 14 (I–VIII). A line that intersects one side of a triangle and a prolongation of another side intersects the third side.

Proof. This simply combines in one statement VIII and theorems 7 and 13.

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* That all of the axioms I–VIII are necessary for the proof of this theorem is demonstrated in §§ 7–14, chap. I.

† This proposition has been regarded as an axiom in previous discussions; cf. § 8, chap. I.

DEF. 5. A point O is in the interior of a triangle if it lies on a segment, the end-points of which are points of different sides of the triangle. The set of such points O is the *interior* of the triangle.

TH. 15 (I–VIII). If O is a point collinear with two points M and N of the perimeter of a triangle ABC, a line joining O to any point P of a side of the triangle meets the perimeter in a point Q $(Q \neq P)$.

Proof. This is easily seen if O lies on one of the lines AB, BC, or CA, or if M or N is a vertex; in the latter case, if MON, by joining OA, OB, and OC.

Thus we may suppose that M and N are interior points of AB and AC respectively. In case of the order MNO (figs. 9 and 10), considering the

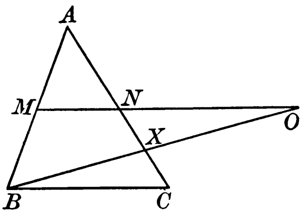

FIG. 9.

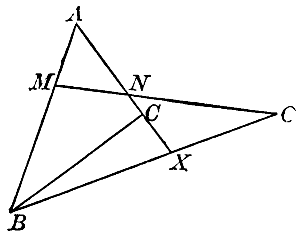

FIG. 10.

triangles OMB, we see that X exists in the orders BXO and ANX, and in view of the order ANC, we have either AXC or ACX. In either case the theorem reduces to theorem 14, in connection with the triangles ABX and BXC.

In case of the order MON (fig. 11), we have by consideration of the triangles MNB and BNC a point X in the order AOX and BXC. Likewise, a point Y in the orders BOY and CYA, and a point Z in the orders COZ and AZB. If P is a point of BC, the theorem follows (14) by consideration of the triangle ABX or BXC. A similar argument holds if P is a point of AB or AC.

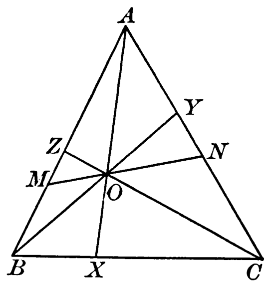

FIG. 11.

DEF. 6. If A, B, C form a triangle, the *plane* ABC consists of all points collinear with any two points of the sides of the triangle.

TH. 16 (I–VIII). Any three non-collinear points lying in a plane determine the plane.

Proof. Let the given plane be ABC. If O is any point of a side AB of the triangle ABC, the points of the plane ABC consist of the points of the lines joining O to the perimeter of ABC. This is a corollary of theorem 15. We prove:

1st, that if A' is any point of the line AB, such that $A' \neq B$, the plane ABC is the same as the plane $A'BC$. In the case of the order $A'AB$ or $AA'B$ (fig. 12), we choose O so that it lies between A or A' and B, and the proposition is an evident consequence of theorem 14. In case of the order ABA', we have just shown that the plane ABC is the same as $AA'C$, which is the same as $BA'C$ or $A'BC$.

2nd, that if C' is any point of the plane ABC not collinear with A and B, the plane ABC is the same as ABC'. Take P between A and B and join

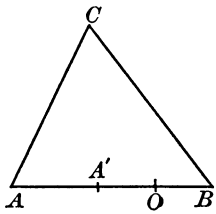

FIG. 12.

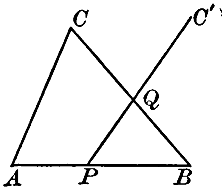

FIG. 13.

PC' (fig. 12), meeting the perimeter of ABC in a point Q, say $Q = C$ or Q on the side BC. By the second part of the proof, ABC is the same as PBC, and hence as PQB, and hence as $PC'B$, and hence as $AC'B$.

3rd, the second step shows that if $A'B'C'$ are any three non-collinear points of the plane, the notation $A'B'C'$ being assigned so that A' is not in BC and B' is not in $A'C$, ABC is the same as $A'BC$, hence the same as $A'B'C$, and hence as $A'B'C'$.

Corollary. A line lying wholly in a plane π and having a point in common with the interior or one side of a triangle which also lies in the plane π meets the perimeter of the triangle in two points.

TH. 17 (I–VIII). A line having two points in common with a plane lies wholly in the plane.

Proof. Let the two points be taken as A and B in defining the plane, ABC. The plane contains the line AB.

Corollary. If two planes have two points in common they have a line in common.

§ 3. *Properties of space.*

AXIOM IX. *If there exist three points not lying in the same line, there exists a plane ABC, such that there is a point D not lying in the plane ABC.*

TH. 18 (I–IX). If ABC is any plane, there exists a point D not in this plane.

Proof. If the plane contained all points it would contain the points A, B,

C mentioned in IX, and would therefore by theorem 16 be the same as the plane ABC of IX, and that plane would contain all points which is contrary to IX.

DEF. 7. If A, B, C, D are four points, not lying in the same plane, they form a *tetrahedron* $ABCD$ whose *faces* are the interiors of the triangles ABC, BCD, CDA, DAB; whose *vertices* are the four points A, B, C, D; and whose *edges* are the six segments AB, BC, CD, DA, AC, BD. A point X is in the *interior* of the tetrahedron if it lies between two points D and E of different faces. The points of the faces, edges, and vertices constitute the *surface* of the tetrahedron.

TH. 19 (I–IX). If any vertex A of a tetrahedron $ABCD$ is joined to an interior point O, the line AO meets the opposite face BCD in a point P lying in the order AOP.

Proof. Since O is an interior point, there exist two points X and Y of the surface in the order XOY (fig. 14). These two points are not both in the same

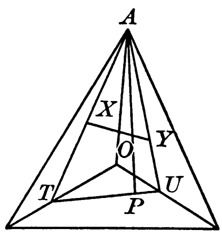

FIG. 14.

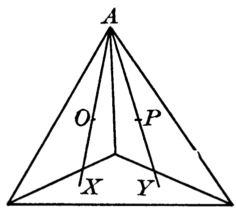

FIG. 15.

face of the tetrahedron, and therefore one of them, X is in a face that has A for a vertex. Join AX, and if Y is in a face that has A for a vertex, join AY also. AX and AY will then meet the perimeter of BCD in two points, T and U. In the triangles XYT and TYU we have the condition of theorem 14. Hence the required point P exists.

If Y is in the face BCD, we have a point T of the perimeter of BCD in the order AXT. Then by theorem 15 U exists on the perimeter of BCD in the order TYU. But in the triangle TXY we again have the hypothesis of theorem 14 and hence a point P in the order TPY. In view of the order TYU, P is in the order TPU.

TH. 20 (I–IX). If any interior point O of a tetrahedron $ABCD$ is joined to another point P of the surface or interior of the tetrahedron, the line OP meets the surface in two points Q and R (where if P is of the perimeter, $P = Q$) lying in the order QOR.

Proof. By theorems 13 and 19, X and Y exist on the interior or perimeter of BCD in the order AOX and APY (fig. 15). Then, by theorem 13, T and U exist on the perimeter of BCD in the orders TXU, and, if $Y \neq U$, TYU. Hence by § 2 OP meets the perimeter of the triangle ATU, and hence the surface of the tetrahedron, in the required points Q and R.

TH. 21 (I–IX). If O is a point in the order MNO with two points M and N of different faces of a tetrahedron $ABCD$, the line joining O to an interior point P of a face of the tetrahedron meets the surface in a point Q, $Q \neq P$.

Proof [I–IX]. 1st (fig. 16). Suppose that P is on the same face with M or N. If with M, let X be a point of this face in the order MPX. Then, (14), OP meets the segment NX in a point Y. Y is an interior point of the tetrahedron, and hence the existence of Q follows from theorem 20. A similar argument holds if P is on the same face with N.

2d (fig. 17). Suppose that P is not on the same face with M or N. Let A be the vertex not in the same face with P. Since A is a vertex of every

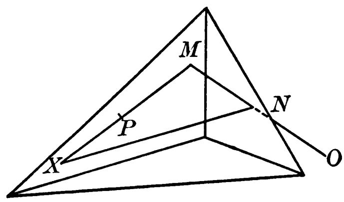

FIG. 16.

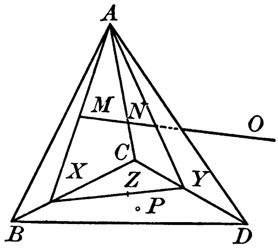

FIG. 17.

face but BCD, on joining AM and AN we obtain X and Y as points of the perimeter of BCD in the orders AMX and ANY. Hence by theorem 15 the line joining O to any point Z in the order XZY meets the perimeter of the triangle AXY in a point W. That PO meets the perimeter now follows from the first part of this proof if O is not in the plane BCD and from § 2 if O is in the plane BCD.

DEF. 8. A *space* $ABCD$, if A, B, C, and D form a tetrahedron, consists of all points collinear with two points of the faces of the tetrahedron.

TH. 22 (I–IX). If two points of a line lie in a space $ABCD$, so does every point of the line.

Proof. If the two points are both complanar with one of the faces of the tetrahedron $ABCD$, apply the results of § 2. If not, let the two points in question be E and F (fig. 18). By theorems 20 and 21, if P is a point of one face, not collinear with E and F, Q and R exist on the surface of $ABCD$, such that Q is on the line PF and R on the line PE, and P, Q, R are not collinear. F and E lie in the plane PQR, and therefore any point of the line X can be joined to two points of the perimeter of the triangle PQR. But every point of PQR is on the surface or in the interior of $ABCD$, and therefore by theorems 20 and 21 every point of the line FE lies in the space $ABCD$.

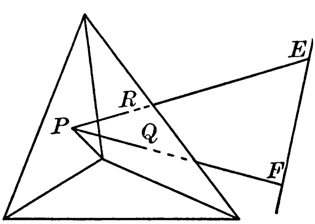

FIG. 18.

TH. 23 (I–IX). Any four non-complanar points of a space define that space.

Proof. Let the given space be $ABCD$, and the four non-complanar points be A', B', C', D'. By theorem 22 every point of the lines $A'B'$ and $A'C'$ belongs to $ABCD$. Hence every point of the plane $A'B'C'$, and hence every point of the space $A'B'C'D'$. Similarly, every point of $ABCD$ belongs to the space $A'B'C'D'$.

Corollary. If three points of a plane lie in a space, so does every point of the plane.

AXIOM X. *If there exist four points neither lying in the same line nor lying in the same plane, there exists a space $ABCD$ such that there is no point E not collinear with two points of the space $ABCD$.*

TH. 24 (I–X). There is but one space.

Proof. By theorem 22 and 23 and axiom X.

TH. 25 (I–X). Two planes that have a point in common have a line in common.

Proof. It is sufficient under the corollary to theorem 17 to show that two planes α and β have two points in common. If O is the given point, let the plane α be defined by points A_1, A_2, A_3, such that O is an interior point of the triangle $A_1A_2A_3$ (this is possible by 16). If B is a point of the second plane not lying in α, let space be defined by the tetrahedron $BA_1A_2A_3$ (fig. 19). If B' is a third point of β the line $B'O$, which lies wholly in β, meets the surface of the tetrahedron (by 24, 20, 21), in another point, B''. If this point B'' is in α, the theorem is proved; if not, join BB''. The line BB'' lies in β and meets one of the lines A_1A_2, A_2A_3, A_3A_1 in a point common to α and β.

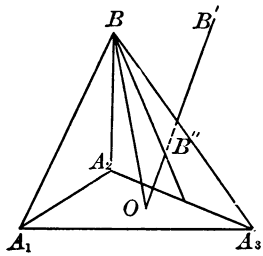

FIG. 19.

§ 4. *Generalizations of order. Regions.*

DEF. 9. The points lying on a system of segments A_1A_2, A_2A_3, $\cdots$, $A_{n-1}A_n$ together with the points (called *vertices*) A_1, A_2, $\cdots$, A_n constitute a *broken line*. If $A_n = A_1$, the broken line, A_1, $\cdots$, A_n is the *boundary of a polygon* of which $A_i (i = 1, \cdots, n-1)$ are the *vertices* and the segments A_iA_{i+1} are the *sides*. A *multiple point* of a polygon or broken line is a point common to a side and a vertex or to two sides or to two vertices. The boundary of a polygon without multiple points is called *simple*.*

DEF. 10. A *region* is a set of points any two of which are points of a broken line composed entirely of points of the set. Two regions, R' and R'', sub-

* A *polygon* is any figure (i. e. class) of sides and vertices, the points of which constitute the "boundary of a polygon." Similar distinctions are made between "angle" and "boundary of an angle," "trihedral angle" and "boundary of a trihedral angle," etc.

sets of the same region, R, are *separated* with respect to R by a set of points $[S]$ if every broken line of R joining a point of R' to a point of R'' contains at least one point of the set $[S]$. $[S]$ is said to *decompose* R into a finite number n of regions if there are n and only n regions R', R'', $\cdots$, $R^{(n)}$ that include every point of R except $[S]$ and if every pair of regions $R^{(i)}$ and $R^{(k)}$ are separated by $[S]$. Lines, planes, and space are obvious examples of regions as is also the interior of a triangle; the latter is referred to as a *triangular region.*

TH. 26 (I–VIII). Any line decomposes a plane in which it lies into two regions.

Proof. Let A be a point of the line a, and A^+, A^- two points of the plane, α, not of a, and in the order A^+AA^-. It is to be proved: (1) that a region including $\frac{A^+}{A^-}$ and including no point of a does not include $\frac{A^-}{A^+}$; (2) that any region of the plane that includes all points that can be joined to a given point by broken lines not meeting a includes either A^+ or A^-.

The statement (1) is a consequence of the proposition that any broken line of the plane joining A^+ to A^- has a point in common with a. If this were not so a certain broken line $A^+B_1B_2\cdots B_nA^-$ would not intersect a. By the corollary to theorem 16, since A^+B_1, B_1, and B_1B_2 have no point in common with a, neither would A^+B_2. By $n-1$ such steps it appears that A^+B_i $(i=2\cdots n)$ would have no point in common with a and hence A^+A^- would not meet a.

The proof of statement (2) depends on the observation that any point P of α but not of a is such that one and only one of the segments PA^+ and PA^- meets a. Otherwise the triangle PA^+A^- would meet the line a only in the point A or else in points of all three sides.

Corollary 1 (I–VIII). A line that meets the boundary of a simple polygon but does not meet any of its vertices meets the boundary of the polygon in an even number of points.

Corollary 2 (I–IX). Any plane α decomposes a space in which it lies into two regions.

DEF. 11. If A and B are two distinct points, the *half-line* AB consists of the segment AB, B, and the prolongation of AB beyond B. If A, B, C are three non-collinear points, the *half-plane* (AB, C) is the one of the two regions, into which the line AB decomposes the plane ABC, that contains C. The two half-planes determined by a line in any plane are called the two *sides* of the line. If A, B, C are non-collinear, the *boundary of the angle* BAC consists of A and the points of the half-lines AB and AC. The *interior of the angle* consists of all points in any region of the plane that does not include the boundary and does include a point X satisfying the order relation PXQ where P is a point of the half-line AB and Q of the half-line AC.

TH. 27 (I–VIII). Two non-intersecting complanar lines (if such exist) decompose their plane into three regions, two intersecting lines decompose it into four regions, and the boundary of an angle decomposes it into two regions.

Proof. Call the two regions into which a line decomposes a plane $+$ and $-$; and then apply the method of theorem 26 to the $+$ and $-$ regions separately. (The new regions will be conveniently designated by $++$, $+-$, $-+$, $--$.)

Corollary 1. In the case of an angle one of the two regions specified by theorem 27 is the interior of the angle. The boundary of an angle meets the boundary of any polygon in its plane and having no vertex or multiple point on the boundary of the angle in an even number of points.

Corollary 2. The boundary of a triangle decomposes the plane in which it lies into two regions, one of which is the interior (def. 5). The other region, which contains lines not intersecting the boundary of the triangle, is called the exterior.

TH. 28 (I–VIII). The boundary of a simple polygon lying entirely in a plane α decomposes α into two regions.

Proof. The proof depends upon two lemmas.

Lemma 1. *If a side of a polygon q intersects a side of a polygon p_n in a single point O not a multiple point of p_n or q, then p_n and q, whether simple or not, have at least one other point in common.* If $n = 3$ (q having any number of sides, m) the theorem reduces to corollary 2, theorem 27. We assume without loss of generality that no three vertices P_{i-1}, P_i, P_{i+1}, are collinear and prove the lemma for every n by reducing to the case $n = 3$. Let p_n have n vertices with the notation such that the side P_1P_2 meets q in the side $Q'_1Q'_2$ where the segment Q'_2O contains no interior point of the triangle $P_1P_2P_3$. By the case $n = 3$, q meets the boundary of the triangle $P_1P_2P_3$ in at least one point other than O. If this point is on the broken line $P_1P_2P_3$ the lemma is verified. If not, q has at least one point on P_1P_3, and at least one of the segments $Q'_1Q'_2$, $Q'_2Q'_3$ has no point or end-point on P_1P_3. Let this segment be one segment of a broken line $Q_kQ_{k+1}\cdots Q_{j-1}Q_j$ of segments of q not meeting P_1P_3 but such that $Q_{k-1}Q_k$ and Q_jQ_{j+1} do each have a point or end-point in common with P_1P_3. ($1 \leqq k < j \leqq m$; if $k = 1$, $Q_{k-1} = Q_m$; if $j = m$, $Q_{j+1} = Q_1$). If O_j is the point common to P_1P_3 and $Q_j\,Q_{j+1}$ or Q_{j+1}, and O_k the point common to P_1P_3 and $Q_{k-1}\,Q_k$ or Q_{k-1}, the broken line $O_kQ_kQ_{k+1}\cdots Q_{j-1}\,Q_jO_j$, has a point inside and also a point outside the triangle $P_1P_2P_3$ and cuts the broken line $P_1P_2P_3$ only once. Hence it has a point inside and a point outside any triangle of which P_1P_3 is a side. On this account if $P_1P_3P_4$ are not collinear, and obviously, if $P_1P_3P_4$ are collinear, q must meet either P_3P_4 or P_4 or P_4P_1. If q does not meet P_3P_4 or P_4, we proceed with $P_1P_4P_5$ as we did with $P_1P_3P_4$. Continuing this process, we either verify the lemma or come by $n-2$ steps to the triangle $P_1P_{n-1}P_n$ and find that q must intersect the broken line $P_{n-1}P_nP_1$, which also verifies the lemma.

Lemma 2. A point P of the boundary of a polygon p_n lying in a plane α is said to be *accessible* from a point O of α if there exists a broken line joining O to P and not meeting the boundary of the polygon in any point other than P. *Every point of the boundary of a simple polygon p_n is accessible from every point O of α, not on the boundary of p_n.* That there is at least one point of p_n accessible from O is seen by considering any line through O meeting the boundary of p_n but not meeting a vertex of p_n. This line meets p_n in a finite number of points and hence in a point P' of a side P_jP_{j+1} such that the segment OP' contains no point of p_n.

If P' is any point of a side P_iP_{i+1} of P_n accessible from O by means of a broken line $OQ_1Q_2\cdots Q_nP'$, it is evident that P_i is accessible from O by means of a similar broken line. For the lines joining P_i to each of the other vertices of p_n meet Q_nP' in at most a finite number of points. There is, therefore, a point Q' such that the segment $Q'P'$ contains none of these points and thus the segment joining P_i to a point of $Q'P'$ does not meet p_n in other points than P_i.

Considerations similar to those just adduced show that if P'' is any point of a side P_iP_{i+1} such that P_i is accessible from O, P'' is also accessible. Now since P' of P_jP_{j+1} is accessible from O, so is every point of P_jP_{j+1} and also P_{j+1}, and hence every point of $P_{j+1}P_{j+2}$, and so on for the whole polygon.

If p_n is any polygon of α, let A and B be two points such that the segment AB intersects a side P_1P_2 but does not meet any other point of the boundary of p_n. By lemma 1, every broken line joining A to B meets p_n. Hence the boundary of any plane polygon whatever decomposes its plane into at least two regions.

If the boundary of a *simple* polygon p_n should decompose α into three regions, let P be a point of the side P_1P_2. P being accessible from each region, there are three segments belonging to different regions whose mutual end-point is P. Two of these segments PC and PD must lie in the same one of the two half-planes defined by P_1P_2. C' and D' being points of PC and PD such that no line joining two vertices of p_n meets PC' or PD', it is evident that $C'D'$ does not meet p_n and hence C' and D' belong to the same region, contradicting the hypothesis of three regions. The boundary of a simple polygon therefore decomposes a plane in which it lies into two and only two regions.

The proofs of the following theorems of this section are similar to the preceding, and therefore are omitted.

DEF. 12. If A, B, C, D are four non-complanar points, the *boundary of the dihedral angle* of the half-planes (AB, C) and (AB, D) consists of the line AB and the half-planes (AB, C) and (AB, D). If AB, AC, AD are three non-complanar half-lines, the *boundary of the trihedral angle* (A, BCD) consists of *the point* A (the *vertex*), the half lines AB,

AC, AD (the *edges*) and the interiors of the angles BAC, CAD. DAB (the *faces*).

TH. 29 (I–IX). Two non-intersecting planes (if such exist) decompose a space in which they lie into three regions, two intersecting planes decompose it into four regions, and a dihedral angle or a trihedral angle decomposes it into two regions.

DEF. 13. The *interior of a dihedral angle* is that region of the two defined by theorem 29 which includes a point X in the order PXQ where P is a point of one of the half-planes of the boundary and Q a point of the other. The *interior of a trihedral angle* (A, BCD) is that of the two regions defined by theorem 29 which includes a point X in the order PXQ where P is a point of the edge AB and Q of the side CAD. The *exterior* of a dihedral or trihedral angle is that one of the two regions defined by theorem 29 which is not interior.

TH. 30 (I–IX). A plane through the vertex and an interior point of a trihedral angle meets the boundary in two half-lines of which one may be an edge and the other interior to a side or each may be interior to a side different from the side to which the other is interior.

DEF. 14. Let a finite or infinite set of complanar lines $\cdots a \cdots b \cdots c \cdots d \cdots k \cdots$ pass through a point O. Let M and N be two points of any line a in the order MON. Let C be any point of a line c. If the lines $\cdots b \cdots$ meet MC in a set of points $\cdots B \cdots$ in the order $M \cdots B \cdots C$ and the lines $\cdots d \cdots k \cdots$ meet CN in a set of points $\cdots D \cdots K \cdots$ in the order $C \cdots D \cdots K \cdots N$, then the set of lines is in the order $a \cdots b \cdots c \cdots k \cdots$. In the special case of the order $abcd$, a and c are said to separate b and d.

TH. 31 (I–VIII). The above definition is independent of the triangle MNP. The notation can be permuted cyclically or reversed. From $a \cdots b \cdots c \cdots k$ and $abkl$, follows $a \cdots b \cdots c \cdots kl$. If a and c separate b and d, a and b do not separate c and d nor do a and d separate b and c. From $abcd$ and $abce$ $(d \neq e)$ follows either $abcde$ or $abced$. In case of the order $abcd$, any segment joining a point of a to a point of c meets one or other of the lines b and d. Any segment joining a point of a to a point of b either meets both c and d or neither.

DEF. 15. Let a finite or infinite set of planes $\alpha \cdots \beta \cdots \gamma \cdots \lambda$ pass through a line o. Intersect o by a plane π. Then if a is the intersection of α and π, b of β and π, etc., and the lines $a \cdots b$, etc., are in the order $a \cdots b \cdots c \cdots l \cdots$, the planes are in the order $\alpha \cdots \beta \cdots \gamma \cdots \lambda \cdots$.

TH. 32 (I–IX). The above definition is independent of the plane π and analogous statements to those of theorem 30 hold for the planes α, β, etc.

§ 5. *Continuity.*

AXIOM XI. *If there exists an infinitude of points, there exist a certain pair of points AC such that if $[\sigma]$ is any infinite set of segments of the line AC, having the property that each point which is A, C, or a point of the segment AC is a point of a segment σ, then there is a finite sub-set $\sigma_1, \sigma_2, \cdots, \sigma_n$ with the same property.*

TH. 33 (I–VIII, XI). If PQ is any segment and $[\sigma]$ any infinite set of segments of the line PQ, having the property that each point which is P, Q, or a point of the segment PQ is an interior point of a segment σ, then there is a finite subset $\sigma_1, \sigma_2, \cdots, \sigma_n$ having the same property.

Proof. In case of the order $APQC$ consider $A'P$ and QC' (where A' and C' are in the order $A'ACC'$) as segments of $[\sigma]$ and the theorem is a corollary of XI.

In case PQ is not on the line AC but $P = A$, let O be a point in the order OCQ. If $[R]$ is the set of those end points of segments σ that lie between P and Q, by § 3 each of the lines OR meets AC in a point B. Thus every segment σ which lies between P and Q corresponds to a segment σ' between A and C. To any segment σ whose end points $S_1 S_2$ are in the order PS_1QS_2 let correspond $D_1D_2 = \sigma'$ where D_1 is the point of intersection of OS_1 with AC and D_2 any point in the order ACD_2. Similarly in the case QS_1PS_2.

By XI there is a finite subset of $[\sigma']$, $\sigma'_1, \sigma'_2, \cdots, \sigma'_n$ of which A, C, and every point of AC are interior points. Any set of σ's corresponding to $\sigma'_1, \cdots, \sigma'_n$ is the required set $\sigma_1, \cdots, \sigma_n$.

If $P = A$ and Q is any point of the line AC let Q' be any point not on the line AC. The theorem holds for PQ' by the preceding paragraph and hence by a similar argument for PQ. If $P \neq Q$ and PQ is any segment not of the line AC, the theorem follows by the intermediation of the segment AP.

TH. 34 (I–VIII, XI). If a segment AB consists of two sets of points $[P]$ and $[Q]$ such that no point P is between two points Q and no point Q between two points P then there exists one and only one point O in the order POQ for every pair of points P and Q $(P \neq O,\ Q \neq O)$.

Proof. Supposing the conclusion of the theorem not valid assign the notation so that there is the order $APQB$; then including A there is a segment not including any point Q and including B there is a segment not including any point P; including every point P there is a segment including no point Q and including every point Q a segment including no point P. Of this set of segments $[\sigma]$ there is (33) a finite sub-set including A, B, and every point of AB. Assign notation to those end points of segments of this set which include points P so that there is the order $A_1A_2 \cdots A_m AP_1P_2 \cdots P_n$; and to those end points of segments of this set which include points Q so that there is the order $Q_k \cdots Q_2Q_1BB_l \cdots B_1$.

By hypothesis we have

$$A_1 A_2 \cdots A_m A P_1 P_2 \cdots P_n Q_k Q_{k-1} \cdots Q_2 Q_1 B B_l \cdots B_1.$$

One of the set of segments whose $n + m + k + l$ end points are here enumerated must include points between P_n and Q_k but such a segment includes both points of P and of Q, contrary to the hypothesis about $[\sigma]$.

That there is only one point O is shown by remarking that if there were two points, O and O', in the order $POO'Q$ for every P, Q there would (6) be a point S in the order OSO' and hence in the orders PSO' and OSQ. But S must be either a P or a Q. In the first case the hypothesis would be contradicted by OSQ and in the second case by PSO'.

DEF. 16. A point P is a *limit point* of a set of points $[S]$ of a line AB if every segment of AB which contains P contains some point of $[S]$ distinct from P.

The following two theorems can be proved either directly from theorem 33 or in the well-known way from theorem 34.

TH. 35 (I–VIII, XI). Every infinite set of points lying on a segment has at least one limit point.

TH. 36 (I–VIII, XI). If A and C are any two points and a set of points $[B]$ is in the order ABC there is a point P in the order ABP for every $B \neq P$, and such that if Q is any point in the order ABQ for every B, either $P = Q$ or APQ.

TH. 37 (I–VIII, XI). If all the lines through a point C, in a plane α, consist of two sets $[s]$ and $[t]$ such that each set contains at least two lines, and no two lines of one set separate any two lines of the other set, then there are two lines o_1 and o_2 that separate every s $(\neq o_1, \neq o_2)$ from every t $(\neq o_1, \neq o_2)$.

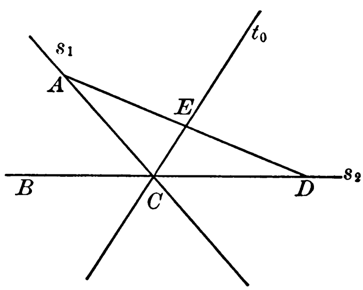

FIG. 20.

Proof. Let s_1 and s_2 be two lines of $[s]$, and A any point of s_1, while B and D are points of s_2 in the order BCD. By § 4, supposing that some t intersects AD there is no t that intersects AB. Let E be the point of intersection of some line t_0 with AD. The theorem follows at once as a corollary of theorem 34, by considering the two sets of points of intersection of $[s]$ and $[t]$ with the segments AE and ED respectively.

§ 6. *Parallel lines.*

TH. 38 (I–VIII, XI). If a is any line of a plane α, through any point C of α there is at least one line of α that does not intersect a.

TH. 39 (I–VIII, XI). If there is more than one such line, there are two lines, c_1 and c_2, through C, not meeting a that separate the lines that do not intersect a from those that do intersect a.

Proof of 38.* Let Q and R be any two points of a, and let $[s]$ be the set of all lines through C that meet a in points S in the order QRS. Let $[t]$ be the set of all other lines through C. No two lines s separate two lines t; for in that case one of the lines t would intersect a on the prolongation of QR beyond R. Hence (37) there are two lines b_1 and b_2 that separate every line s from every line t. Not both of these lines meet a. For if this were so, call their points of intersection B_1 and B_2 and the points of intersection of an s line, S, and of a t line, T. If there is a point T in the order B_1TB_2, there are points S' and S'' in the order $S'B_1B_2S''$, and hence a T point between two S points,—a contradiction with the hypothesis that all points of the prolongation of QR are S points. If there is a point S in the order B_1SB_2, there are points T' and T'' in the order $T'B_1B_2T''$ and hence in the order $T'ST''$ and therefore a T point on the prolongation of QR beyond R.

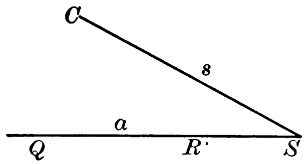

FIG. 21.

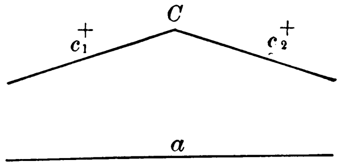

FIG. 22.

Proof of 39. Let all the lines through C that meet a be $[s]$ and all that do not $[t]$. Then the existence of c_1 and c_2 follows at once from theorem 37. c_1 or c_1 cannot intersect a; for in that case we should have $S_1C_1S_2$ or $S_1C_2S_2$.

DEF. 17. In case there are two lines c and c_1 the line a lies wholly in one of the four regions into which the plane is decomposed by c_1 and c_2. Call the two half-lines, c_1^+ and c_2^+, that bound this region of the plane the *half-lines parallel to a*. a and the two parallel half-lines c_1^+ and c_2^+ separate from the rest of the plane a region of the plane which lies *between* them and consists of all points that can be joined by a segment not meeting a or c_1^+ or c_2^+ to a point B in the order CBQ. In case the lines c_1 and c_2 coincide in a line c, that line is called *the parallel to a*.

TH. 40 (I–IX). If a has two parallel half-lines, any line through a point between a and its parallel half-lines, c_1^+ and c_2^+, meets either a or c_1^+ or c_2^+.

Proof. Let O be a point not in the plane α. The planes Oc_1 and Oc_2 and Oa meet in three lines, OC, a_1, and a_2. Let the line through a point between a and c_1^+ and c_2^+ be b. Then, the plane Ob, passing through the interior of the trihedral angle whose vertex is O, meets two of the sides or one side and one edge.

* That XI is essential for this theorem is proved by K_{XI}, § 4, chap. I.

Hence there is at least one line o' common to the plane Ob and a side of the trihedral angle, and by the lemma below, o' meets a or c_1^+ or c_2^+.

Lemma. If o' is any line through O in the face Oa of the trihedral angle, o' meets a; for the plane Co' meets α in a line a that passes through a point in the interior of the angle between c_1^+ and c_2^+. If o'' is any line through O in the face Oc_1^+, it meets c_1^+; for joining o'' to a point C' of the half-line of c_2 other than c_2^+, this plane meets Oa in a line o' that must intersect a in a point A. As the points A and C are on opposite sides of c_1^+, the plane $o''C'$ meets c_1^+, and therefore o'' meets c_1^+.

Th. 41 (I–IX). If in a plane α through a point C there is only one line c parallel to a, any line of α that meets one of the lines a and c meets the other.

Proof. It follows as in the lemma to theorem 40 that if O is a point outside α, and o the intersection of Oc and Oa, any line of one of these planes that passes through O and is distinct from o meets a if it is in Oa and c if it is in Oc.

If a' is a line through a point A of a the plane Oa' meets Oc in a line o' distinct from o, and o' meets c by the first paragraph. Similarly, if c' is a line through a point C' of c, the plane Oc' meets Oa in a line o' distinct from o, and hence o' meets a.

Th. 42 (I–IX, XI). If a is a line of a plane α and through one point C, not on a, there is only one line c that does not intersect a, then through any point C' of α not on a there is only one parallel to a.

Proof. Suppose that through a point $\bar{C}$ not between a and c there are two parallels c_1 and c_2 to a. These lines are also parallel to c: for by theorem 41 every line through $\bar{C}$ that meets c also meets a, and every line that meets a meets c. Then either c lies between a and c_1^+ and c_2^+, or a lies between c_1^+ and c_2^+ and c. Either case is in contradiction with theorem 40. Hence there can be only one parallel through $\bar{C}$. The theorem now follows for any point $\bar{\bar{C}}$ between a and c by considering with a a parallel, on the opposite side of a from $\bar{\bar{C}}$.

Axiom XII. *If a is any line of any plane α there is some point C of α through which there is not more than one line of the plane α which does not intersect a.*

Th. 43 (I–IX, XI, XII). In any plane α through any point A there is one and only one line parallel to a given line a.

Proof. By XII and theorems 38 and 42.

CHAPTER III.

Projective Geometry.

§ 1. *Preliminary theorems.*

The development of projective geometry from the foundation established in the preceding chapter §§ 1–6, follows methods that are fairly well known.

On this account and for lack of space the proofs of most of the theorems will be omitted. While it will not often be possible to give a reference which furnishes complete details of a demonstration, the citation will generally cover the methods.

Projective geometry is independent of the parallel axiom and so in this chapter no use will be made of § 6, chapter II before § 7 where the euclidean metrical geometry is defined. In the present § 1 are stated some preliminary theorems from which are derived, in the following § 2, definitions of the projective elements analogous to those originally proposed by KLEIN.* In neither of these sections is any use made of the continuity axiom XI.

Lemma. Let π and ρ be two planes intersecting in a line a, and let l and m be two lines of ρ lying on the same side of a. Then if l is complanar with each of two lines b and c of π, and m is complanar with b, it follows that m is complanar also with c.

TH. 44 (I–X). If three lines, a, b, and c, of a plane π, are each complanar with a line l not of the plane π, and a and b are each complanar with another line m, then c is also complanar with m.

TH. 45 (I–X). If a and b are two lines in a plane π, and l and m two lines not in π but each complanar with a and b, then l and m are complanar.

DEF. 18. Two lines a and b lying in the same plane π, define a system of lines consisting of every line of intersection of a plane through a with a plane through b as well as all the lines in π that are complanar with one of the lines of the system that does not lie in π. Such a system of lines is called a *bundle*. The system of all lines in a plane π complanar with a line l not in π is called a *pencil*.

TH. 46 (I–X). Every two lines of a bundle are complanar and define the same bundle. Through any point of space passes one line of a given bundle. Any two points B and C not in the same line of a bundle A define with A a plane including all lines of A passing through points of the line BC. Two distinct bundles have in common at most one line.

DEF. 19. If A and B are two bundles, through every point O of space there passes one line of each bundle. If these lines do not coincide they define a plane. The system of planes thus defined by two bundles is called a *pencil of planes*. Every line of A or B not common to A and B lies in one and only one plane of the pencil. A bundle every one of whose lines lies in a plane of the pencil is said to be *incident with* the pencil.

A special case of a pencil of planes is the set all of planes through a line.

TH. 47 (I–X). If $A_1 B_1 C_1$ and $A_2 B_2 C_2$ are two triangles of a plane π, such

* F. KLEIN, Mathematische Annalen, vol. 6 (1873), p. 112. Further details are given by M. PASCH, l. c., pages 40 to 72, and by F. SCHUR, *Ueber die Einführung der sogenannten idealen Elemente in die projective Geometrie,* Mathematische Annalen, vol. 39 (1891), p. 113. A very elegant exposition is due to R. BONOLA, *Sulla introduzione degli enti improprii in geometria projettiva,* Giornale di Matematiche, vol. 38 (1900), p. 105.

that the three bundles defined by the three pairs of sides A_1B_1 and A_2B_2, B_1C_1 and B_2C_2, C_1A_1 and C_2A_2, each has a line in a certain plane σ, $(\sigma \neq \pi)$ then the joints of corresponding vertices, A_1A_2, B_1B_2, C_1C_2, lie in the same pencil.

TH. 48 (I–X.) If a pencil of planes defined by two bundles L_1 and L_2 meets a plane π in three lines, a, b, and c, these three lines lie in the same pencil of lines.

Corollary 1. Any bundle having a line in each of two planes of a pencil is incident with the pencil.

Corollary 2. Any two bundles incident with a pencil of planes determine that pencil.

TH. 49 (I–X). If ABC and $A_1B_1C_1$ are two triangles of a plane π such that the lines AA_1, BB_1, CC_1, are in the same pencil, then the bundles defined by pairs AB, A_1B_1, BC, B_1C_1, and AC, A_1C_1, lie in the same pencil of planes.

Corollary. If two triangles ABC and $A'B'C'$ are so situated in a plane that the lines joining their corresponding vertices AA', BB', CC', meet in a point P, and the corresponding sides AB, $A'B'$, etc., meet in three points, C_1, C_2, C_3 then C_1, C_2, C_3 are collinear; and conversely.

TH. 50 (I–X). If three bundles, A, E, and C lie in one pencil of planes, whereas three bundles B, C, and D lie in another pencil of planes, then there is a bundle F incident at once with the pencils AB and DE.

§ 2. *The projective elements.*

DEF. 20. A *projective point* is a bundle of lines.

DEF. 21. A *projective line* is a pencil of planes. Three points incident with the same line are collinear.

DEF. 22. A *projective plane* is the set of all projective points collinear with the projective point O and any point of a projective line o not incident with O (including O and the points incident with o). Any point of the plane is said to be *incident with* the plane.

DEF. 23. A projective point is called *proper* if the bundle of lines defining it meet in a point. A projective line or plane is proper if there is incident with it a proper point; otherwise it is *improper* or *ideal.*

It will immediately become evident that a proper projective point has the same incidence relations as a point in the sense of chapter II. For the rest of this chapter we shall mean by "point" a projective point, and by "proper point" a proper projective point.

By corollary 2, theorem 48, any two points incident with a line determine that line. Let a plane π be determined by a line a and point B not incident with a. That any other point and line incident with π determine it, is an immediate deduction from theorem 38. This amounts to saying that any three points incident with π determine it, and that any line, two of whose points are incident with π is also incident with π. It follows that any two lines incident with π are incident with one, and only one, common point.

A line incident with a proper point is by def. 23 and by theorem 25 incident with another proper point, and hence the ordinary line defined by these points is contained in the projective line. Consequently, every plane incident with a with a proper point is incident with three non-collinear proper points, and therefore the ordinary (chapter II) plane determined by these three points is included in the projective plane.

By def. 21, every two proper planes are incident with one, and only one line, proper or improper. A proper plane λ and an improper μ are incident with one and only one improper line. For let m be any line of μ; it is by definition the intersection of two proper planes α and β through two proper points A and B of λ that are therefore incident with two proper lines a and b of λ. a and b define a point at once incident with m and λ. Hence every line m of μ is incident with a point L of λ. But as three non-collinear points determine a plane, these points L must be incident with a line, which proves our proposition.

Finally, two improper planes, ρ and σ, are incident with one and only one line. For a proper plane α is incident with one line of each of them, so that the point incident with each of these lines is incident with each of the planes ρ and σ. Another proper plane β gives another point, and just as in the preceding paragraph, these two points determine the one and only one line incident with ρ and σ.

As obvious results of the above argument, we have:*

TH. 51 (I–X). Two points are incident with one and only one line.

Two planes are incident with one and only one line.

TH. 52 (I–X). Three points not incident with the same line are incident with one and only one plane.

Three planes not incident with the same line are incident with one and only one point.

TH. 53 (I–X). A line and a point not incident with it are incident with one and only one plane.

A line and a plane not incident with it are incident with one and only one point.

TH. 54 (I–X). Two lines incident with the same plane are incident with one and only one point.

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* No attempt is here made to obtain an *independent* system of basal theorems for projective geometry. The foundations of projective geometry as a science by itself have been studied by M. PIERI, *Sui principii che reggono la geometria della retta*, Atti della Reale Accademia della Scienze di Torino, vol. 36 (1901), p. 335. *I principii della geometria di posizione* ···, Memorie della Reale Accademia della Scienze di Torino, vol. 48 (1899), p. 1.

Two lines incident with the same point are incident with one and only one plane.

TH. 55 (I–X). If two points are incident with a plane, any line incident with each of them is incident with the plane.

If two planes are incident with a point, any line incident with each of them is incident with the point.

The terms pencil and bundle, which were used of the elements of chapter I, will now be applied in an analogous way to the projective elements.

DEF. 24. The set of all lines incident at once with any plane α and any point A (incident with α) is called a *pencil* of lines.

The set of all points incident with a line a is called a *pencil* of points.

The set of all planes incident with a line a is called a *pencil* of planes or *axial pencil.*

Pencils of lines, points, and planes are called *one-dimensional forms* or *forms of the first grade.*

DEF. 25. A *two-dimensional form* or *form of the second grade* is one of the following:

1. The set of all lines incident with a point.
2. The set of all lines incident with a plane.
3. The set of all planes incident with a point.
4. The set of all points incident with a plane.

1 and 3 are also called *bundles of lines* and *bundles of planes.* The system of all points and lines incident with a plane is called a *plane system* or *field;* and the set of all planes and lines incident with a point is called a *point system* or *point field.*

DEF. 26. The set of all points in space and the set of all planes in space are called *three dimensional forms* or *forms of the third grade.* The set of all points and planes in space is the space system or space field.

DEF. 27. A *figure* is any set of points, lines, and planes.

DEF. 28. If A is a point incident with no point of a figure w (consisting of points and lines) the system of all elements at once incident with A and the elements of w is the *projection* of w from A. A is called a *center of perspectivity.* If w' is a figure consisting of lines and planes, and a (or α) a line (or plane) not incident with any element of w', then the system of all elements at once incident with a (or α) and w' is the *section* of w' by a (or α). If two figures have a projection or section in common, or if one is a section of the other, they are said to be *perspective* figures.

DEF. 29. If $1, 2, 3, \cdots$ are elements of a one-dimensional form, and $1', 2', 3', \cdots$ are elements of another one-dimensional form, perspective with $1, 2, 3, \cdots$, we write $(1, 2, 3, \cdots) \,\overline{\wedge}\, (1', 2', 3', \cdots)$. If

$$(1, 2, 3, \cdots) \,\overline{\wedge}\, (1', 2', 3', \cdots) \cdot \cdot \,\overline{\wedge}\, (1^n, 2^n, 3^n, \cdots) \quad (n=2, 3, \cdots),$$

we say that the figures $(1, 2, 3, 4, \cdots)$ and $(1^n, 2^n, 3^n, 4^n, \cdots)$ are *projective*, and use the notation $(1, 2, 3, 4, \cdots) \barwedge (1^n, 2^n, 3^n, 4^n, \cdots)$.

§ 3. *Order and continuity of projective elements.*

DEF. 30. A finite or infinite set of collinear points are in the *order* $A \cdots B \cdots C \cdots K$ if they are projected from a proper point O by lines in the order $a \cdots b \cdots c \cdots k$ (see chapter I, § 4).

Lemma *a*. This definition is independent of the choice of O.

Lemma *b*. If two sets of points A, B, C, $\cdots$ and $A'B'C' \cdots$ are sections of the same (one-dimensional) pencil of lines in such a way that A, A', B, B', etc., are incident with the same line of the pencil, then if one set is in the order $A \cdots B \cdots C \cdots K$, the other set is in the order $A' \cdots B' \cdots C' \cdots K'$.

Lemma *c*. If two ranges l_1 and l_2 are sections of a pencil of planes S and the points A_1 and A_2, B_1 and B_2, etc., are incident with the same plane, then if the order of a set of points on l_1 is $A_1 \cdots B_1 \cdots C_1 \cdots K_1$, the order of the corresponding set on l_2 is $A_2 \cdots B_2 \cdots C_2 \cdots K_2$.

DEF. 31. A set of elements of any form of the first grade (not a pencil of points) are in the order $\cdots l \cdots m \cdots n \cdots k \cdots$ if their section by a line is a set of points in the order $\cdots L \cdots M \cdots N \cdots K \cdots$. If four elements are in the order 1 2 3 4, 1 and 3 are said to *separate* 2 and 4.

TH. 56 (I–X). If 1 2 3 are any three distinct elements of a one-dimensional form, there exists an element 4 in the order 1 2 3 4. If elements 1, 2, 3, 4 are in the order 1 2 3 4 they are distinct. The notation for the order of a set of elements $\cdots l \cdots n \cdots m \cdots k \cdots$ of a one-dimensional form can be permuted cyclically or reversed. No other permutation is possible. From $l \cdots n \cdots m \cdots k$ and $lnkp$ follows $l \cdots n \cdots m \cdots kp$. From 1 2 3 4 and 1 2 3 5 $(4 \neq 5)$ follows either 1 2 4 5 or 1 2 5 4. Any set of elements $\cdots l' \cdots k' \cdots$ perspective or projective with a set in the order $\cdots l \cdots n \cdots m \cdots k \cdots$ in such a way that l' corresponds to l, etc., are in the order $\cdots l' \cdots n' \cdots m' \cdots k' \cdots$.

TH. 57 (I–XI). If all the elements of a one-dimensional form consist of two sets, such that each set consists of at least two elements, and no two elements of one set separate two elements of the other set, then there exist two elements p and p' that separate each element (distinct from p and p') of one set from each element (distinct from p and p') of the other set.

§ 4. *Principle of duality.*

All the theorems of projective geometry can be deduced from theorems 51–57. These theorems remain entirely unchanged if the words point and plane are interchanged. Hence anything that can be deduced from them about points can be deduced also about planes. Hence we have, referring to a proposition deducible from theorems 51–57 as a projective proposition:

TH. 58 (I–XI). *Fundamental theorem of duality.* Any valid projective proposition remains valid if the words point and plane are interchanged in its complete statement.

From this duality of space follows a like duality among the planes and lines or points and lines of any point- or plane-field. If π is any plane, any theorem about the incidence and order of the lines and points in π will hold for the incidence and order of the planes and lines obtained by projecting the lines and points of π from a point O. The space dual of this latter theorem is a theorem about points and lines, in any plane σ; or in particular about the plane $\sigma = \pi$. Hence we have:

TH. 59 (I–XI). Any valid projective proposition about points and lines incident with a plane π remains valid if the words *point* and *line* are interchanged.

As the space dual of theorem 59 we have:

TH. 60 (I–XI). Any valid projective proposition about lines and planes incident with a point P remains valid if the words *plane* and *line* are interchanged.

The general term incidence has been used in this section to bring out the duality of space in a formal way. In the following sections, however, we shall use the words intersect, lie in, are on, etc., according to the ordinary usage.

§ 5. *Harmonic conjugates.*

The following theorems are well-known and are satisfactorily proved in the standard text-books.

TH. 61 (I–X). If A, B, C are three points of a line l, and A', B', C' are any three points of any other line l', ABC can be projected to $A'B'C'$ by the use of two centers of perspectivity.

Corollary. If A, B, C, and A', B', C' are on the same line l, three centers of perspectivity are sufficient.

TH. 62 (I–X). If A, B, C, D are any four points of a line,

$$(ABCD) \barwedge (BADC) \barwedge (CDAB).$$

DEF. 32. The figure consisting of three non-collinear points and the three lines incident with them by pairs is called a *triangle.*

DEF. 33. The figure consisting of four complanar points (no three of which are collinear) and the six lines incident with them by pairs is called a *complete quadrangle.* The four points are called the *vertices;* the six lines, the *sides;* two sides that do not have a vertex in common are *opposite sides;* and the points of intersection of opposite sides are *diagonal points.* The triangle of the diagonal points is called the *diagonal triangle.*

DEF. 34. The plane dual of the above figure is a *complete quadrilateral.* It has four sides, six vertices, and three diagonal lines, etc.

TH. 63 (I–X). Let ABC and $A'B'C'$ be two triangles lying in the same or in different planes. If the joins of corresponding vertices, AA', BB', CC', meet in a point, the corresponding sides, AB with $A'B'$, BC with $B'C'$, AC with $A'C'$, intersect in three points of a line. Conversely, if the intersections of corresponding sides lie on a line the joins of corresponding vertices meet in a point.

TH. 64 (1–X). If five sides of one complete quadrangle intersect five sides of another complete quadrangle (no point of intersection being a vertex) in collinear points, then the point of intersection of the sixth side of one with the sixth side of the other lies on the same line with the other five intersection points.

DEF. 35. If A and C are diagonal points of a quadrangle, and B and D the intersections of the remaining pair of opposite sides with the line AC, D is called the fourth harmonic or harmonic conjugate of B with respect to A and C.

TH. 65 (I–X). If D is the harmonic conjugate of B with respect to A and C, there is only one such point D, B is the harmonic conjugate of D with respect to A and C, and the points A, C are separated by B, D.

DEF. 36. If D is the fourth harmonic of B with respect to A and C, the pair BD are said to separate AC harmonically; $ABCD$ are sometimes said to be four harmonic points.

TH. 66 (I–X). If $(ABCD) \barwedge (A'B'C'D')$ and the pair BD separates the pair AC harmonically, then $B'D'$ separates $A'C'$ harmonically. *Conversely*, if B and D separate A and C harmonically, and A', B', C', D' are any four points such that B', D' separate A', C' harmonically, then $(ABCD) \barwedge (A'B'C'D')$. If B and D separate A and C harmonically, then A and C separate B and D harmonically.

Corollary. If $ABCD$ are four harmonic points, besides the perspectivities of theorem 62 $(ABCD) \barwedge (ADCB) \barwedge (CBAD)$.

§ 6. *The fundamental theorem of projective geometry.*

DEF. 37. If A, B, C are any three points of a line, a point D is said to be harmonically related to A, B, C if it is any one of a finite set of points $D_1 \cdots D_n$ ($n \geqq 1$) such that D_1 is the fourth harmonic of one of A, B, C with respect to the other two and D_k ($k = 2 \cdots n$) the fourth harmonic of one of the set $ABCD_1 \cdots D_{k-1}$ with respect to two others of the set.

TH. 67 (I–XI). If A, B, C are any three points of a line and P and Q any two other points of the same line then there are two points D and E which at the same time are harmonically related to A, B, C and separate P and Q.*

* F. KLEIN (Mathematische Annalen, vol. 6 (1873) p. 139) first pointed out that VON STAUDT's proof of the fundamental theorem was incomplete without the use of some such axiom as XI.

TH. 68 (I–XI). *The fundamental theorem.* If $ABCD$ are four collinear points and $(ABCD) \barwedge (A'B'C'D')$ then by any process of projection and section for which $(ABCD) \barwedge (A'B'C'D'')$, $D' = D''$.*

TH. 69 (I–XI) If A, B and C, D are two pairs of points that do not separate one another then there exists one and only one pair P, Q that at once harmonically separates A, B and C, D.

By means of this theorem it is seen that any one to one correspondence preserving harmonic relations also preserves order and then by theorem 67 and the principle of duality we obtain the following general form of the "fundamental theorem."

DEF. 38. A one-to-one correspondence between two forms (or of a form with itself) is called projective if, whenever elements $1, 2, 3, \cdots$ of the same one-dimensional form correspond to elements $1', 2', 3', \cdots$,

$$(123\cdots) \barwedge (1'2'3'\cdots).$$

TH. 70 (I–XI). *A more general fundamental theorem.* Any one-to-one correspondence between two one-dimensional forms by which every four harmonic elements of one form correspond to four harmonic elements of the other form, is projective.

TH. 71 (I–XI). A one-to-one correspondence between two forms I and I′ (where I may be the same as I′) of the same grade higher than the first is projective if every one-dimensional form of elements of I corresponds to a to a one-dimensional form of I′, and reciprocally.

DEF. 39. A projective correspondence by which points correspond to points is a *collineation.* A projective correspondence by which elements of one kind correspond to elements of another kind is called *polar.*

§ 7. *Involution and polar system.*

DEF. 40. An *involution* is such a projective correspondence of a one-dimensional form with itself that if an element 1 corresponds to 1′ then 1′ corresponds to 1. Pairs of elements such as 1 and 1′ are called pairs of *conjugate* elements. The following theorems stated for pencils of points apply by duality to involutions in general.

TH. 72 (I–XI). An involution has either two or no self-corresponding points.

M. PASCH (Vorlesungen über Neuere Geometrie, § 15, Leipzig, 1882) gave a proof which made use of congruence axioms and the so-called Archimedean axiom.

H. WIENER (*Ueber die Grundlagen und den Aufbau der Geometrie,* Jahresbericht der Deutschen Mathematikervereinigung, vol. 1, p. 47, vol. 3, p. 70) and SCHUR (Mathematische Annalen, vol. 51 (1898), p. 401, vol. 55 (1902), p. 265) have shown that it can be demonstrated by congruence axioms alone without the Archimedean axiom.

L. BALSER (Mathematische Annalen, vol. 55 (1902), p. 293) has given a proof based upon hypotheses equivalent to ours.

Corollary. If an involution has two fixed points, A, B, they separate harmonically every pair of corresponding points.

TH. 73 (I–XI). If A, A', B, B' are arbitrary points of a line, there exists one and only one involution in which A and A', B and B' are conjugate points.

Corollary. The theorem is true also if $A = A'$ or $B = B'$, or both.

TH. 74 (I–XI). The three pairs of opposite sides of a complete quadrangle $QRST$ are cut in three pairs of an involution by any straight line which lies in the plane of the quadrangle and passes through none of its vertices.

TH. 75 (I–XI). If two pairs, AA', BB', of an involution do not separate each other, no two pairs of the involution separate each other, and the involution has two fixed points. If two pairs, AA', BB', separate each other, every two pairs of the involution separate each other, and the involution has no fixed point.

DEF. 41. An involution with two fixed points is called *hyperbolic*. An involution with no fixed point is called *elliptic*.

TH. 76 (I–XI). If B and C are two points of a line, there is one, and only one, pair of a given elliptic involution that separates B and C harmonically.

TH. 77 (I–XI). If two pencils are projective, an involution in one corresponds to an involution in the other.

DEF. 42. The points and lines of a plane constitute a *polar system* if they are in such a reciprocal one to one correspondence that to the $\substack{\text{point}\\ \text{line}}$ incident with any two $\substack{\text{lines}\\ \text{points}}$ corresponds the $\substack{\text{line}\\ \text{point}}$ incident with the corresponding pair of $\substack{\text{points}\\ \text{lines}}$. A point is the *pole* of its corresponding line, and a line the *polar* of its corresponding point. A polar system is *elliptic* if no element is incident with its corresponding element; it is *hyperbolic* if some element is incident with its corresponding element. A $\substack{\text{line } l'\\ \text{point } L'}$ is *conjugate* to a $\substack{\text{line } l\\ \text{point } L}$ if it passes through the $\substack{\text{pole}\\ \text{polar}}$ of $\substack{l\\ L}$.

The proofs of the existence of the various collineation and polar systems depend on constructions that are well-known and for which it is easily verified that our hypotheses are sufficient. See

J. STEINER: *Synthetische Geometrie, Die Theorie der Kegelschnitte*, Leipzig, 1887.

H. SCHROETER: *Theorie der Oberflächen Zweiter Ordnung*, Leipzig, 1880.

K. G. C. VON STAUDT: *Beiträge zur Geometrie der Lage*, Nürnberg, 1856.

In particular, in STEINER'S *Synthetische Geometrie* on pp. 411–414 and 422–424 will be found the proof of the existence of elliptic polar systems.

In the following section we outline the definition and deduction of metric euclidean geometry. For that argument we employ the following theorems:

TH. 78 (I–XI). In any polar system, if a $\substack{\text{line } l \\ \text{point } L}$ is conjugate to a $\substack{\text{line } l' \\ \text{point } L'}$, $\substack{l' \\ L'}$ is conjugate to $\substack{l \\ L}$. The system of all such pairs of conjugate $\substack{\text{lines} \\ \text{points}}$ of a given $\substack{\text{point} \\ \text{line}}$ form an involution. There exist polar systems in which all these involutions are elliptic. A polar system is fully determined by the involutions upon a pair of conjugate $\substack{\text{points} \\ \text{lines}}$.

TH. 79 (I–XI). If $A_1 A_2 A_3$ are the vertices of a triangle and $P_1 P_2 P_3$ the points of intersection of the lines $A_2 A_3$, $A_3 A_1$, $A_1 A_2$ respectively with a line p, and $M_1 M_2 M_3$ the harmonic conjugates of $P_1 P_2 P_3$ with respect to $A_2 A_3$, $A_3 A_1$, $A_1 A_2$ then the three lines joining $M_1 M_2 M_3$ to the three conjugate points of $P_1 P_2 P_3$ in any involution on p, meet in a point.

The metric special case of this theorem is that the perpendiculars at the middle points of the sides of a triangle meet in a point. For its proof one has only to consider the complete quadrangle $PM_1 M_2 M_3$ where P is the intersection of $P_1' M_1$ with $P_2' M_2$, P_1' and P_2' being the conjugate points of P_1 and P_2. It leads at once to the following

TH. 80 (I–XI). Let Σ be an elliptic polar system and $A_1 A_2 A_3$ any triangle and $M_1 P_1$, $M_2 P_2$ the unique pairs of conjugate points that separate $A_2 A_3$ and $A_3 A_1$ harmonically. If P_3 is the point of intersection of $P_1 P_2$ (or of $M_1 M_2$) with $A_1 A_2$ then the fourth harmonic M_3 of P_3 with respect to $A_1 A_2$ is also its conjugate with respect to Σ. Furthermore the three conjugate lines of the sides of the triangle at M_1, M_2, M_3 meet in a point P, the pole of $P_1 P_2$.

§ 7. *Similar figures.**

DEF. 43. In this section the term *projective transformation* will be used as synonymous with *collineation*. If a point X corresponds to a point X' in a collineation, X is said to be projectively *transformed into* or to *go into*, X'. A *reflection* of the points of a line by a pair of points of the line AA' is a projective transformation by which each point of the line goes into its fourth harmonic with respect to A and A'; i. e., the pairs of corresponding points form an hyperbolic involution; For this transformation we use the notation (AA').

A *reflection* of the points of a $\substack{\text{plane} \\ \text{space}}$ by the (non-incident) point A and $\substack{\text{line } a \\ \text{plane } \alpha}$ of the $\substack{\text{plane} \\ \text{space}}$ is a projective transformation such that each point X of the $\substack{\text{plane} \\ \text{space}}$ goes into its fourth harmonic with respect to A and the intersection of AX with $\substack{a \\ \alpha}$. We use the notation $\substack{(Aa) \\ (A\alpha)}$.

If by one projective transformation, μ, points A, B, C, D, $\cdots$ go into $A'B'C'D' \cdots$ and by another, ν, $A'B'C'D' \cdots$ go into $A''B''C''D'' \cdots$, then the correspondence of $ABCD \cdots$ with $A''B''C''D'' \cdots$ is evidently projective; it is called the *product* of ν and μ and written $\nu\mu$. We write $\mu A = A'$ and thus

* For references see footnote, ††, § 1, chap. I.

have $\nu(\mu A) = \nu A' = A'' = (\nu\mu)A$. Further $\rho(\nu\mu) = (\rho\nu)\mu$ and if σ is any reflection σ^2 is the identity. We denote the identity by ω — thus $\sigma^2 = \omega$.

If a transformation is such that every pair of conjugate elements of an involution or polar system, Σ, goes into the same or another conjugate pair, the transformation is said to *leave* Σ *invariant.*

TH. 81 (I–XI).* Any collineation that leaves invariant an elliptic involution or a (plane) elliptic polar system is either a reflection by a pair of corresponding elements or a product of such reflections. A reflection by any pair of an elliptic involution or polar system leaves the involution or polar system invariant. By a suitably chosen transformation leaving the system invariant, any pair of incident elements of the system can be transformed into any other pair of incident elements of the same kind.

Combining results of this theorem with the theory of parallel lines (§ 6, chapter II) we are able to give a definition of congruent angles, and thus to establish the theory of similar figures. It results immediately from theorem 43 (the strong parallel proposition) that the improper projective points constitute an improper plane, whereas every other projective point is a bundle of lines, all passing through a point in the sense of chapter I. The improper plane (whose determination is unique by theorem 43) we call the plane at infinity. In this plane an elliptic polar system † Σ is arbitrarily chosen. Σ determines at every point of space a polar system of planes and lines, of which a corresponding plane and line (or pair of conjugate lines or pair of conjugate planes) are said to be mutually perpendicular. In formal terms:

DEF. 44. A plane α and line a are mutually *perpendicular* if their intersections with the plane at infinity are a pair of corresponding elements a' and A' of Σ. Two intersecting lines are perpendicular if they meet the plane at infinity in conjugate points of Σ.

DEF. 45. One angle is *congruent* to another if the sides of the one can be transformed into the sides of the other by a collineation of space that leaves Σ invariant. Since this relation is evidently mutual the angles are said to be congruent to one another.

The properties of congruent angles follow without difficulty from this definition and from theorem 81. This may be conveniently verified by comparison with HILBERT's axioms IV 4 and IV 5.

Our axioms and definitions are therefore sufficient to establish the theory of similar figures.

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* There is, of course, an analogous theorem for space polar systems.

† "The imaginary circle at infinity." That the choice of Σ is arbitrary is one of the important properties of space; one tends to overlook this if congruence is introduced by axioms.

§ 8. *Congruent segments.*

To define congruent segments we have only to specify a subgroup of the group of similarity collineations of space that leave Σ (and hence, of course, the plane at infinity) invariant. A perpendicular reflection or more simply (since we shall no longer deal with other reflections) a reflection, by a proper plane α, is the reflection $(A\alpha)$ where A is the correspondent in Σ of the line a in which α meets the plane at infinity.

DEF. 46. A segment AB is *congruent* to a segment $A'B'$ if AB can be transformed into $A'B'$ by a finite number of reflections. Since this relation is evidently mutual the segments are said to be congruent to one another.

That the usual properties of congruent segments can be deduced from this definition is obvious on comparison with HILBERT's axioms IV 1, 2, 3, 6, as soon as one establishes, by the aid of thorems 79 and 80:

TH. 82 (I–XII). OA is the only segment with end-point O, in the half-line OA, that is congruent to OA.

§ 9. *The system of axioms is categorical.*

The metric properties of space having been established as indicated in § 8, the usual definitions can be given for the length of a segment in terms of a unit segment.

Coördinate axes can now be introduced in the usual way: namely, a unit of length, and three non-complanar lines through a point O are chosen arbitrarily. On each line there is thus established a correspondence between the points and the real numbers, positive and negative. The coördinates of a point P are defined as the numbers associated with the points in which three planes through the point P and parallel to the three pairs of coördinate lines meet the corresponding three coördinate lines.

TH. 83 (I–XII). To every set of three coördinates (x, y, z) corresponds a point, and to every point corresponds a set of three coördinates. The necessary and sufficient condition that three distinct points, P_1, P_2, P_3, whose coördinates are respectively (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) are in the order $P_1P_2P_3$, is the existence of a number k such that $0 < k < 1$, and every one of the following fractions, not of the form $\frac{0}{0}$, is equal to k.

$$\frac{x_1 - x_2}{x_1 - x_3}, \quad \frac{y_1 - y_2}{y_1 - y_3}, \quad \frac{z_1 - z_2}{z_1 - z_3}.$$

TH. 84 (I–XII). If K and K' are any two classes that verify axioms I–XII, then any proposition stated in terms of points and order that is valid of the class K is valid of the class K'.

Proof. In K and in K' introduce a system of coördinates; let a point in one class correspond to a point in the other class whose coördinates are the same three numbers. By theorem 83, if three points of K are in a given order, the corresponding three points of K' are in the same order.